

ON THE LARGE TIME BEHAVIOR OF SOLUTIONS OF HAMILTON-JACOBI EQUATIONS ASSOCIATED WITH NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. In this article, we study the large time behavior of solutions of first-order Hamilton-Jacobi Equations, set in a bounded domain with nonlinear Neumann boundary conditions, including the case of dynamical boundary conditions. We establish general convergence results for viscosity solutions of these Cauchy-Neumann problems by using two fairly different methods : the first one relies only on partial differential equations methods, which provides results even when the Hamiltonians are not convex, and the second one is an optimal control/dynamical system approach, named the “weak KAM approach” which requires the convexity of Hamiltonians and gives formulas for asymptotic solutions based on Aubry-Mather sets.

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INTRODUCTION

We are interested in this article in the large time behavior of solutions of first-order Hamilton-Jacobi Equations, set in a bounded domain with nonlinear Neumann boundary conditions, including the case of dynamical boundary conditions. The main originality of this paper is twofold : on one hand, we obtain results for these nonlinear Neumann type problems in their full generality, with minimal assumptions (at least we think so) and, on the other hand, we provide two types of proofs following the two classical approaches for these asymptotic problems : the first one by the PDE methods which has the advantages of allowing to treat cases when the Hamiltonians are non-convex, the second one by an optimal control/dynamical system approach which gives a little bit more precise description of the involved phenomena. For Cauchy-Neumann problems with linear Neumann boundary conditions, the asymptotic behavior has been established very recently and independently by the second author in [23] by using the dynamical approach and the first and third authors in [5] by using the PDE approach.

In order to be more specific, we introduce the following initial-boundary value problems

$$(CN) \quad \begin{cases} u_t + H(x, Du) = 0 & \text{in } \Omega \times (0, \infty), \\ B(x, Du) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \overline{\Omega} \end{cases}$$

and

$$(DBC) \quad \begin{cases} u_t + H(x, Du) = 0 & \text{in } \Omega \times (0, \infty), \\ u_t + B(x, Du) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain of \mathbb{R}^n with a C^1 -boundary and u is a real-valued unknown function on $\overline{\Omega} \times [0, \infty)$. We, respectively, denote by $u_t := \partial u / \partial t$ and $Du := (\partial u / \partial x_1, \dots, \partial u / \partial x_n)$ its time derivative and gradient with respect to the space variable. The functions $H(x, p), B(x, p)$ are given real-valued continuous function on $\overline{\Omega} \times \mathbb{R}^n$; more precise assumptions on H and B will be given at the beginning of Section 1.

Throughout this article, we are going to treat these problems by using the theory of viscosity solutions and thus the term “viscosity” will be omitted henceforth. We also point out that the boundary conditions have to be understood in the viscosity sense: we refer the reader to the “User’s guide to viscosity solutions” [9] for a precise definition which is not recalled here.

The existence and uniqueness of solutions of (CN) or (DBC) are already well known. We refer to the articles [1, 2, 3, 20] and the references therein.

The standard asymptotic behavior, as $t \rightarrow +\infty$, for solutions of Hamilton-Jacobi Equations is the following : the solution $u(x, t)$ is expected to look like $-ct + v(x)$ where the constant c and the function v are solutions of an *additive eigenvalue* or *ergodic problem*. In our case, we have two different

ergodic problems for (CN) and (DBC): indeed, looking for a solution of the form $-at + w(x)$ for (CN), where a is constant and w a function defined on $\overline{\Omega}$, leads to the equation

$$(E1) \quad \begin{cases} H(x, Dw(x)) = a & \text{in } \Omega, \\ B(x, Dw(x)) = 0 & \text{on } \partial\Omega \end{cases}$$

while, for (DBC), the function w has to satisfy

$$(E2) \quad \begin{cases} H(x, Dw(x)) = a & \text{in } \Omega, \\ B(x, Dw(x)) = a & \text{on } \partial\Omega. \end{cases}$$

We point out that one seeks, here, for a pair (w, a) where $w \in C(\overline{\Omega})$ and $a \in \mathbb{R}$ such that w is a solution of (E1) or (E2). If (w, a) is such a pair, we call w an *additive eigenfunction* or *ergodic function* and a an *additive eigenvalue* or *ergodic constant*.

A typical result, which was first proved for Hamilton-Jacobi Equations set in \mathbb{R}^n in the periodic case by P.-L. Lions, G. Papanicolaou and S. R. S. Varadhan [26], is that there exists a unique constant $a = c$ for which this problem has a *bounded* solution, while the associated solution w may not be unique, even up to an additive constant. This non-uniqueness feature is a key difficulty in the study of the asymptotic behavior.

The main results of this article are the following : under suitable (and rather general) assumptions on H and B

- (i) There exists a unique constant c such that (E1) (resp., (E2)) has a solution in $C(\overline{\Omega})$.
- (ii) If u is a solution of (CN) (resp., (DBC)), then there exists a solution $(v, c) \in C(\overline{\Omega}) \times \mathbb{R}$ of (E1) (resp., (E2)), such that

$$u(x, t) - (v(x) - ct) \rightarrow 0 \quad \text{uniformly on } \overline{\Omega} \text{ as } t \rightarrow \infty. \quad (0.1)$$

The rest of this paper consists in making these claims more precise by providing the correct assumptions on H and B , by recalling the main existence and uniqueness results on (CN) and (DBC), by solving (E1) and (E2), and proving (i) and finally by showing the asymptotic result (ii). In an attempt to make the paper concise, we have decided to present the full proof of (ii) for (CN) only by the optimal control/dynamical system approach while we prove the (DBC) result only by the PDE approach. To our point of view, these proofs are the most relevant one, the two other proofs following along the same lines and being even simpler.

In the last decade, the large time behavior of solutions of Hamilton-Jacobi equation in compact manifold \mathcal{M} (or in \mathbb{R}^n , mainly in the periodic case) has received much attention and general convergence results for solutions have been established. G. Namah and J.-M. Roquejoffre in [30] are the first to prove (0.1) under the following additional assumption

$$H(x, p) \geq H(x, 0) \text{ for all } (x, p) \in \mathcal{M} \times \mathbb{R}^n \text{ and } \max_{\mathcal{M}} H(x, 0) = 0, \quad (0.2)$$

where \mathcal{M} is a smooth compact n -dimensional manifold without boundary. Then A. Fathi in [12] proved the same type of convergence result by dynamical systems type arguments introducing the “weak KAM theory”. Contrarily to [30], the results of [12] use strict convexity (and smoothness) assumptions on $H(x, \cdot)$, i.e., $D_{pp}H(x, p) \geq \alpha I$ for all $(x, p) \in \mathcal{M} \times \mathbb{R}^n$ and $\alpha > 0$ (and also far more regularity) but do not require (0.2). Afterwards J.-M. Roquejoffre [31] and A. Davini and A. Siconolfi in [11] refined the approach of A. Fathi and they studied the asymptotic problem for Hamilton-Jacobi Equations on \mathcal{M} or n -dimensional torus. The second author, Y. Fujita, N. Ichihara and P. Loreti have investigated the asymptotic problem specially in the whole domain \mathbb{R}^n without the periodic assumptions in various situations by using the dynamical approach which is inspired by the weak KAM theory. See [13, 21, 16, 17, 18]. The first author and P. E. Souganidis obtained in [7] more general results, for possibly non-convex Hamiltonians, by using an approach based on partial differential equations methods and viscosity solutions, which was not using in a crucial way the explicit formulas of representation of the solutions. Later, by using partially the ideas of [7] but also of [30], results on the asymptotic problem for unbounded solutions were provided in [6].

There also exists results on the asymptotic behavior of solutions of convex Hamilton-Jacobi Equation with boundary conditions. The third author [27] studied the case of the state constraint boundary condition and then the Dirichlet boundary conditions [28, 29]. J.-M. Roquejoffre in [31] was also dealing with solutions of the Cauchy-Dirichlet problem which satisfy the Dirichlet boundary condition pointwise (in the classical sense) : this is a key difference with the results of [28, 29] where the solutions were satisfying the Dirichlet boundary condition in a generalized (viscosity solutions) sense. These results were slightly extended in [5] by using an extension of PDE approach of [7].

We also refer to the articles [31, 8] for the large time behavior of solutions to time-dependent Hamilton-Jacobi equations. Recently E. Yokoyama, Y. Giga and P. Rybka in [32] and the third author with Y. Giga and Q. Liu in [14, 15] has gotten the large time behavior of solutions of Hamilton-Jacobi equations with noncoercive Hamiltonian which is motivated by a model describing growing faceted crystals. We refer to the article [10] for the large-time asymptotics of solutions of nonlinear Neumann-type problems for viscous Hamilton-Jacobi equations.

This paper is organized as follows: in Section 1 we state the precise assumptions on H and B , as well as some preliminary results on (CN), (DBC), (E1) and (E2). Section 2 is devoted to the proof of convergence results (0.1) by the PDE approach. In Section 3 we devote ourselves to the proof of convergence results (0.1) by the optimal control/dynamical system approach. Then we need to give the variational formulas for solutions of (CN) and (DBC) and results which are related to the weak KAM theory, which are new and interesting themselves. In Appendix we give the technical lemma which is used in Section 2 and the proofs of basic results which are presented in Section 1.

Before closing the introduction, we give a few comments about our notation. We write $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$ for $x \in \mathbb{R}^n$, $r > 0$ and $B_r := B_r(0)$. For $A \subset \mathbb{R}^l$, $B \subset \mathbb{R}^m$ for $l, m \in \mathbb{N}$ we denote by $C(A, B)$, $LSC(A, B)$, $USC(A, B)$, $Lip(A, B)$ the space of real-valued continuous, lower semicontinuous, upper semicontinuous, Lipschitz continuous and on A with values in B , respectively. For $p \in \mathbb{R}$ we denote by $L^p(A, B)$ and $L^\infty(A, B)$ the set of all measurable functions whose absolute value raised to the p -th power has finite integral and which are bounded almost everywhere on A with values in B , respectively. We write $C^k(A)$ for the sets of k -th continuous differentiable functions for $k \in \mathbb{N}$. For given $-\infty < a < b < \infty$ and $x, y \in B$, we use the symbol $AC([a, b], B)$ to denote the set of absolutely continuous functions on $[a, b]$ with values in B . We call a function $m : [0, \infty) \rightarrow [0, \infty)$ a modulus if it is continuous and nondecreasing on $[0, \infty)$ and vanishes at the origin.

1. PRELIMINARIES AND MAIN RESULT

In this section, we introduce the key assumptions on H, B and we present basic PDE results on (CN) and (DBC) (existence, comparison,..., etc.) which will be used throughout this article. The proofs are given in the appendix.

We use the following assumptions.

(A0) Ω is a bounded domain of \mathbb{R}^n with a C^1 -boundary.

In the sequel, we denote by $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a C^1 -defining function for Ω , i.e. a C^1 -function which is negative in Ω , positive in the complementary of $\overline{\Omega}$ and which satisfies $D\rho(x) \neq 0$ on $\partial\Omega$. Such a function exists because of the regularity of Ω . If $x \in \partial\Omega$, we have $D\rho(x)/|D\rho(x)| = n(x)$ where $n(x)$ is the unit outward normal vector to $\partial\Omega$ at x . In order to simplify the presentation and notations, we will use below the notation $\tilde{n}(x)$ for $D\rho(x)$, even if x is not on $\partial\Omega$. Of course, if $x \in \partial\Omega$, $\tilde{n}(x)$ is still an outward normal vector to $\partial\Omega$ at x , by assumption $\tilde{n}(x)$ does not vanish on $\partial\Omega$ but it is not anymore a unit vector.

(A1) The function H is continuous and coercive, i.e.,

$$\lim_{r \rightarrow \infty} \inf \{H(x, p) : x \in \overline{\Omega}, |p| \geq r\} = \infty.$$

(A2) For any $R > 0$, there exists a constant $M_R > 0$ such that

$$|H(x, p) - H(x, q)| \leq M_R |p - q|$$

for all $x \in \overline{\Omega}$ and $p, q \in B_R$.

(A3) There exists $\theta > 0$ such that

$$B(x, p + \lambda \tilde{n}(x)) - B(x, p) \geq \theta \lambda$$

for all $x \in \partial\Omega$, $p \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ with $\lambda \geq 0$.

(A4) There exists a constant $M_B > 0$ such that

$$|B(x, p) - B(x, q)| \leq M_B |p - q|$$

for any $x \in \partial\Omega$ and $p, q \in \mathbb{R}^n$.

(A5) The function $p \mapsto B(x, p)$ is convex for any $x \in \partial\Omega$.

We briefly comment these assumptions. Assumption (A1) is classical when considering the large time behavior of solutions of Hamilton-Jacobi Equations since it is crucial to solve ergodic problems. Assumption (A2) is a non-restrictive technical assumption while (A3)-(A4) are (almost) the definition of a nonlinear Neumann boundary condition. Finally the convexity assumption (A5) on B will be necessary to obtain the convergence result. We point out that the requirements on the dependence of H and B in x are rather weak : this is a consequence of the fact that, because of (A1), we will deal (essentially) with Lipschitz continuous solutions (up to a regularization of the subsolution by sup-convolution in time. Therefore the assumptions are weaker than in the classical results (cf. [1, 2, 3, 20]).

A typical example for B is the boundary condition arising in the optimal control of processes with reflection which has control parameters:

$$B(x, p) = \sup_{\alpha \in \mathcal{A}} \{ \gamma_\alpha(x) \cdot p - g_\alpha(x) \},$$

where \mathcal{A} is a compact metric space, $g_\alpha : \partial\Omega \rightarrow \mathbb{R}$ are given continuous functions and $\gamma_\alpha : \overline{\Omega} \rightarrow \mathbb{R}^n$ is a continuous vector field which is oblique to $\partial\Omega$, i.e.,

$$\tilde{n}(x) \cdot \gamma_\alpha(x) \geq \theta$$

for any $x \in \partial\Omega$ and $\alpha \in \mathcal{A}$.

Our first result is a comparison result.

Theorem 1.1 (Comparison Theorem for (CN) and (DBC)). *Let $u \in \text{USC}(\overline{\Omega} \times [0, \infty))$ and $v \in \text{LSC}(\overline{\Omega} \times [0, \infty))$ be a subsolution and a supersolution of (CN) (resp., (DBC)), respectively. If $u(\cdot, 0) \leq v(\cdot, 0)$ on $\overline{\Omega}$, then $u \leq v$ on $\overline{\Omega} \times [0, \infty)$.*

Then, applying carefully Perron's method (cf. [19]), we have the existence of Lipschitz continuous solutions.

Theorem 1.2 (Existence and Regularity of Solutions of (CN) and (DBC)). *For any $u_0 \in C(\overline{\Omega})$, there exists a unique solution $u \in \text{UC}(\overline{\Omega} \times [0, \infty))$ of (CN) or (DBC). Moreover, if $u_0 \in W^{1,\infty}(\Omega)$, then u is Lipschitz continuous on $\overline{\Omega} \times [0, \infty)$ and therefore u_t and Du are uniformly bounded. Finally, if u and v are the solutions which are respectively associated to u_0 and v_0 , then*

$$\|u - v\|_{L^\infty(\Omega \times (0, \infty))} \leq \|u_0 - v_0\|_{L^\infty(\Omega)}. \quad (1.1)$$

Finally we consider the additive eigenvalue/ergodic problems.

Theorem 1.3 (Existence of Solutions of (E1) and (E2)). *There exists a solution $(v, c) \in W^{1,\infty}(\Omega) \times \mathbb{R}$ of (E1) (resp., (E2)). Moreover, the additive eigenvalue is unique and is represented by*

$$c = \inf \{ a \in \mathbb{R} : \text{(E1) (resp., (E2)) has a subsolution} \}. \quad (1.2)$$

The following proposition shows that, taking into account the ergodic effect, we obtain bounded solutions of (CN) or (DBC). This result is a straightforward consequence of Theorems 1.3 and 1.1.

Proposition 1.4 (Boundedness of Solutions of (CN) and (DBC)). *Let c be the additive eigenvalue for (E1) (resp., (E2)). Let u be the solution of (CN) (resp., (DBC)). Then $u + ct$ is bounded on $\bar{\Omega} \times [0, \infty)$.*

From now on, replacing $u(x, t)$ by $u(x, t) + ct$, we can normalize the additive eigenvalue c to be 0. As a consequence H is also replaced by $H - c$ and B by $B - c$ in the (DBC)-case. In order to obtain the convergence result, we use the following assumptions.

(A6) Either of the following assumption (A6)₊ or (A6)_− holds.

(A6)₊ There exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p + q) \geq \eta$ and $H(x, q) \leq 0$ for some $x \in \bar{\Omega}$ and $p, q \in \mathbb{R}^n$, then for any $\mu \in (0, 1]$,

$$\mu H(x, \frac{p}{\mu} + q) \geq H(x, p + q) + \psi_\eta(1 - \mu).$$

(A6)_− There exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p + q) \leq -\eta$ and $H(x, q) \leq 0$ for some $x \in \bar{\Omega}$ and $p, q \in \mathbb{R}^n$, then for any $\mu \geq 1$,

$$\mu H(x, \frac{p}{\mu} + q) \leq H(x, p + q) - \frac{\psi_\eta(\mu - 1)}{\mu}.$$

In the optimal control/dynamical system approach, the following assumptions are used

(A7) The function H is convex, i.e., for each $x \in \Omega$ the function $p \mapsto H(x, p)$ is convex on \mathbb{R}^n and either of the following assumption (A7)₊ or (A7)_− holds.

(A7)_± There exists a modulus ω satisfying $\omega(r) > 0$ for $r > 0$ such that for any $(x, p) \in \bar{\Omega} \times \mathbb{R}^n$, if $H(x, p) = c$, $\xi \in \partial_p H(x, p)$ and $q \in \mathbb{R}^n$, then

$$H(x, p + q) \geq c + \xi \cdot q + \omega((\xi \cdot q)_\pm).$$

We point out that we use the notation $\partial_p H(x, p)$ for the convex subdifferential of the function $p \mapsto H(x, p)$ where x is fixed.

Our main result is the following theorem.

Theorem 1.5 (Large-Time Asymptotics). *Assume (A0)-(A6) or (A0)-(A5) and (A7). For any $u_0 \in C(\bar{\Omega})$, if u is the solution of (CN) (resp., (DBC)) associated to u_0 , then there exists a solution $v \in W^{1,\infty}(\Omega)$ of (E1) (resp., (E2)), such that*

$$u(x, t) \rightarrow v(x) \quad \text{uniformly on } \bar{\Omega} \text{ as } t \rightarrow \infty.$$

Remark 1.1. (i) Under the convexity assumption on H of (A7), the assumptions (A1)–(A6) are equivalent to (A1)–(A5) and (A7), and therefore (A1)–(A5) and (A7) imply (A1)–(A6). Indeed, under (A1) and the convexity assumption on H , conditions (A7)_± are equivalent to (A6)_±, respectively. This equivalence in the plus case has been proved in [16, Appendix C]. The proof in the minus case is similar to that in the plus case, which we leave to the

interested reader. (ii) We notice that if H is smooth with respect to the p -variable, then (A6) is equivalent to a *one-sided directionally strict convexity* in a neighborhood of the level set $\{p \in \mathbb{R}^n : H(x, p) = 0\}$ for all $x \in \mathbb{T}^n$, i.e.,

(A6') *there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p+q) \geq \eta$ and $H(x, q) \leq 0$ (or if $H(x, p+q) \leq -\eta$ and $H(x, q) \leq 0$) for some $x \in \mathbb{T}^n$ and $p, q \in \mathbb{R}^n$, then for any $\mu \in (0, 1]$,*

$$D_p H(x, p+q) \cdot p - H(x, p+q) \geq \psi_\eta.$$

(iii) Let us take the Hamiltonian $H(x, p) := (|p|^2 - 1)^2$ for instance. If we consider the homogeneous Neumann condition, then we can easily see that the additive eigenvalue is 1. This Hamiltonian is not convex but satisfies (A6) (and (A6')).

2. ASYMPTOTIC BEHAVIOR I : THE PDE APPROACH

As we mentioned it in the introduction, we provide the proof of Theorem 1.5 only in the case of the nonlinear dynamical-type boundary value problem (DBC). In the case of the nonlinear Neumann-type boundary condition, the proof is simpler and we will only give a remark at the end of this section.

In order to avoid technical difficulties, we assume that u_0 is Lipschitz continuous (and therefore the solution u of (DBC) is Lipschitz continuous on $\overline{\Omega} \times [0, \infty)$). We can easily remove it by using (1.1).

As in [7, 5] the *asymptotic monotonicity* of solutions of (DBC) is a key property to get convergence (0.1).

Theorem 2.1 (Asymptotic Monotonicity).

(i) **(Asymptotically Increasing Property)** *Assume that (A6)₊ holds. For any $\eta \in (0, \eta_0]$, there exists $\delta_\eta : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{s \rightarrow \infty} \delta_\eta(s) \rightarrow 0$ and*

$$u(x, s) - u(x, t) + \eta(s - t) \leq \delta_\eta(s)$$

for all $x \in \overline{\Omega}$, $s, t \in [0, \infty)$ with $t \geq s$.

(ii) **(Asymptotically Decreasing Property)** *Assume that (A6)₋ holds. For any $\eta \in (0, \eta_0]$, there exists $\delta_\eta : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{s \rightarrow \infty} \delta_\eta(s) \rightarrow 0$ and*

$$u(x, t) - u(x, s) - \eta(t - s) \leq \delta_\eta(s)$$

for all $x \in \overline{\Omega}$, $s, t \in [0, \infty)$ with $t \geq s$.

The **proof of Theorem 1.5** follows as in [7, 5]: we reproduce these arguments for the convenience of the reader.

Since $\{u(\cdot, t)\}_{t \geq 0}$ is compact in $W^{1, \infty}(\Omega)$, there exists a sequence $\{u(\cdot, T_n)\}_{n \in \mathbb{N}}$ which converges uniformly on $\overline{\Omega}$ as $n \rightarrow \infty$. Theorem 1.1 implies that we have

$$\|u(\cdot, T_n + \cdot) - u(\cdot, T_m + \cdot)\|_{L^\infty(\Omega \times (0, \infty))} \leq \|u(\cdot, T_n) - u(\cdot, T_m)\|_{L^\infty(\Omega)}$$

for any $n, m \in \mathbb{N}$. Therefore, $\{u(\cdot, T_n + \cdot)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $C(\overline{\Omega} \times [0, \infty))$ and it converges to a function denoted by $u^\infty \in C(\overline{\Omega} \times [0, \infty))$.

Fix any $x \in \overline{\Omega}$ and $s, t \in [0, \infty)$ with $t \geq s$. By Theorem 2.1 we have

$$u(x, s + T_n) - u(x, t + T_n) + \eta(s - t) \leq \delta_\eta(s + T_n)$$

or

$$u(x, t + T_n) - u(x, s + T_n) - \eta(t - s) \leq \delta_\eta(s + T_n)$$

for any $n \in \mathbb{N}$ and $\eta > 0$. Sending $n \rightarrow \infty$ and then $\eta \rightarrow 0$, we get, for any $t \geq s$

$$u^\infty(x, s) \leq u^\infty(x, t).$$

or

$$u^\infty(x, t) \leq u^\infty(x, s).$$

Therefore, we see that the functions $x \mapsto u^\infty(x, t)$ are uniformly bounded and equi-continuous, and they are also monotone in t . This implies that $u^\infty(x, t) \rightarrow w(x)$ uniformly on $\overline{\Omega}$ as $t \rightarrow \infty$ for some $w \in W^{1,\infty}(\Omega)$. Moreover, by a standard stability property of viscosity solutions, w is a solution of (DBC).

Since $u(\cdot, T_n + \cdot) \rightarrow u^\infty$ uniformly in $\overline{\Omega} \times [0, \infty)$ as $n \rightarrow \infty$, we have

$$-o_n(1) + u^\infty(x, t) \leq u(x, T_n + t) \leq u^\infty(x, t) + o_n(1),$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$, uniformly in x and t . Taking the half-relaxed semi-limits as $t \rightarrow +\infty$, we get

$$-o_n(1) + w(x) \leq \liminf_{t \rightarrow \infty} u(x, t) \leq \limsup_{t \rightarrow \infty}^* u(x, t) \leq w(x) + o_n(1).$$

Sending $n \rightarrow \infty$ yields

$$w(x) = \liminf_{t \rightarrow \infty} u(x, t) = \limsup_{t \rightarrow \infty}^* u(x, t)$$

for all $x \in \overline{\Omega}$. And the proof of Theorem 1.5 is complete.

Now we turn to the **proof of Theorem 2.1**.

Noticing that the additive eigenvalue is 0 again, by Proposition 1.4 the solution u of (DBC) is bounded on $\overline{\Omega} \times [0, \infty)$. We consider any solution v of (E2). We notice that $v - M$ is still a solution of (E2) for any constant $M > 0$. Therefore subtracting a positive constant to v if necessary, we may assume that

$$1 \leq u(x, t) - v(x) \leq C \quad \text{for all } (x, t) \in \overline{\Omega} \times [0, \infty) \text{ and some } C > 0 \quad (2.1)$$

and we fix such a constant C .

We define the functions $\mu_\eta^\pm : \overline{\Omega} \times [0, \infty) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mu_\eta^+(x, s) &:= \min_{t \geq s} \left(\frac{u(x, t) - v(x) + \eta(t - s)}{u(x, s) - v(x)} \right), \\ \mu_\eta^-(x, s) &:= \max_{t \geq s} \left(\frac{u(x, t) - v(x) - \eta(t - s)}{u(x, s) - v(x)} \right) \end{aligned} \quad (2.2)$$

for $\eta \in (0, \eta_0]$. By the uniform continuity of u and v , we have $\mu_\eta^\pm \in C(\overline{\Omega} \times [0, \infty))$. It is easily seen that $0 \leq \mu_\eta^+(x, s) \leq 1$ and $\mu_\eta^-(x, s) \geq 1$ for all $(x, s) \in \overline{\Omega} \times [0, \infty)$ and $\eta \in (0, \eta_0]$.

Lemma 2.2. (i) Assume that $(A6)_+$ holds. The function μ_η^+ is a supersolution of

$$\begin{cases} \max\{w - 1, w_t + M|Dw| \\ \quad + \frac{\psi_\eta}{C}(w - 1)\} = 0 & \text{in } \Omega \times (0, \infty), \\ \max\{w - 1, w_t + F(x, Dw)\} = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases} \quad (2.3)$$

for any $\eta \in (0, \eta_0]$ and some $M > 0$, where

$$F(x, p) := -K(-p \cdot n(x)) + M_B|p - (p \cdot n(x))n(x)|, \quad (2.4)$$

and

$$K(r) := \begin{cases} \theta r & \text{if } r \geq 0, \\ M_B r & \text{if } r < 0. \end{cases} \quad (2.5)$$

(ii) Assume that $(A6)_-$ holds. The function μ_η^- is a subsolution of

$$\begin{cases} \min\{w - 1, w_t - M|Dw| \\ \quad + \frac{\psi_\eta}{C} \cdot \frac{w - 1}{w}\} = 0 & \text{in } \Omega \times (0, \infty), \\ \min\{w - 1, w_t - F(x, -Dw)\} = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases}$$

for any $\eta \in (0, \eta_0]$ and some $M > 0$.

Before proving Lemma 2.2 we notice that Lemma 2.2 implies

$$\mu_\eta^+(\cdot, s) \rightarrow 1 \text{ uniformly on } \overline{\Omega},$$

$$\mu_\eta^-(\cdot, s) \rightarrow 1 \text{ uniformly on } \overline{\Omega}$$

as $s \rightarrow \infty$. Indeed noting that $r \mapsto F(x, p + rn(x)) - \min\{\theta, M_B\}r$ is non-decreasing and the function $(r - 1)/r$ is increasing for $r > 0$, we see that the comparison principle holds for both of Neumann problems

$$\begin{cases} \max\{w - 1, M|Dw| + \frac{\psi_\eta}{C}(w - 1)\} = 0 & \text{in } \Omega, \\ \max\{w - 1, F(x, Dw)\} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.6)$$

and

$$\begin{cases} \min\{w - 1, -M|Dw| + \frac{\psi_\eta}{C} \cdot \frac{w - 1}{w}\} = 0 & \text{in } \Omega, \\ \min\{w - 1, -F(x, -Dw)\} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.7)$$

Moreover we have $\liminf_{s \rightarrow \infty} \mu_\eta^+, 1$ are solutions of (2.6) and $\limsup_{s \rightarrow \infty}^* \mu_\eta^-, 1$ are a subsolution and a solution of (2.7), respectively. Therefore from these observations we see $\liminf_{s \rightarrow \infty} \mu_\eta^+ = 1$ and $\limsup_{s \rightarrow \infty}^* \mu_\eta^- = 1$, which imply the conclusion.

Proof of Lemma 2.2. We only prove (i), since we can prove (ii) similarly. Fix $\eta \in (0, \eta_0]$ and let μ_η^+ be the function given by (2.2). We recall that $\mu_\eta^+(x, s) \leq 1$ for any $x \in \overline{\Omega}$, $s \geq 0$.

Let $\phi \in C^1(\overline{\Omega} \times [0, \infty))$ and $(\xi, \sigma) \in \overline{\Omega} \times (0, \infty)$ be a strict local minimum of $\mu_\eta^+ - \phi$, i.e., $\mu_\eta^+(x, s) - \phi(x, s) > \mu_\eta^+(\xi, \sigma) - \phi(\xi, \sigma)$ for all $(x, s) \in \overline{\Omega} \times [\sigma - r, \sigma + r] \setminus \{(\xi, \sigma)\}$ and some small $r > 0$. Since we can get the conclusion by

the same argument as in [7] in the case where $\xi \in \Omega$, we only consider the case where $\xi \in \partial\Omega$ in this proof. Moreover since there is nothing to check in the case where $\mu_\eta^+(\xi, \sigma) = 1$ or $\phi_t(\xi, \sigma) + F(\xi, D\phi(\xi, \sigma)) \geq 0$, we assume that

$$\mu_\eta^+(\xi, \sigma) < 1 \text{ and } \phi_t(\xi, \sigma) + F(\xi, D\phi(\xi, \sigma)) < 0. \quad (2.8)$$

We choose $\tau \geq \sigma$ such that

$$\mu_\eta^+(\xi, \sigma) = \frac{u(\xi, \tau) - v(\xi) + \eta(\tau - \sigma)}{u(\xi, \sigma) - v(\xi)} =: \frac{\mu_2}{\mu_1}.$$

We write μ for $\mu_\eta^+(\xi, \sigma)$ henceforth.

Next, for $\alpha > 0$ small enough, we consider the function

$$(x, t, s) \mapsto \frac{u(x, t) - v(x) + \eta(t - s)}{u(x, s) - v(x)} + |x - \xi|^2 + (t - \tau)^2 - \phi(x, s) + 3\alpha\rho(x),$$

where ρ is the function which is defined just after (A0). We notice that, for $\alpha = 0$, (ξ, τ, σ) is a strict minimum point of this function. This implies that, for $\alpha > 0$ small enough, this function achieves its minimum over $\overline{\Omega} \times \{(t, s) : t \geq s, s \in [\sigma - r, \sigma + r]\}$ at some point $(\xi_\alpha, t_\alpha, s_\alpha)$ which converges to (ξ, τ, σ) when $\alpha \rightarrow 0$. Then there are two cases : either (i) $\xi_\alpha \in \Omega$ or (ii) $\xi_\alpha \in \partial\Omega$. We only consider case (ii) here too since, again, the conclusion follows by the same argument as in [7] in case (i). In case (ii), since $\rho(\xi_\alpha) = 0$, the α -term vanishes and we have $(\xi_\alpha, t_\alpha, s_\alpha) = (\xi, \tau, \sigma)$ by the strict minimum point property.

For any $\delta \in (0, 1)$ let $C_1^{\xi, \delta}$ and $C_2^{\xi, \delta} \in C^1(\mathbb{R}^n)$ be, respectively, the functions given in Lemma 4.2 with $a = \mu_1/\mu, b = -\eta/\mu$ and $a = \frac{\mu_1}{1-\mu}, b = 0$, and let χ_1 and χ_2 be, respectively, the functions given in Lemma 4.3 with $C_{a,b}^{\xi, \delta} = C_1^{\xi, \delta}$ and $C_2^{\xi, \delta}$ for $\varepsilon > 0$. We set $K := \overline{\Omega}^3 \times \{(t, s) : t \geq s, s \in [\sigma - r, \sigma + r]\}$. We define the function $\Psi : K \rightarrow \mathbb{R}$ by

$$\begin{aligned} & \Psi(x, y, z, t, s) \\ &:= \frac{u(x, t) - v(z) + \eta(t - s)}{u(y, s) - v(z)} - \phi(y, s) + \chi_1(x - y) + \chi_2(x - z) \\ &+ |x - \xi|^2 + (t - \tau)^2 - \alpha(\rho(x) + \rho(y) + \rho(z)). \end{aligned}$$

In view of Lemma 4.3, if $A \geq M_2$, where A is the constant in χ_1, χ_2 , then Ψ achieves its minimum over K at some point $(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s})$ which depends on $\alpha, \delta, \varepsilon$. By taking a subsequence if necessary we may assume that

$$\bar{x}, \bar{y}, \bar{z} \rightarrow \xi \text{ and } \bar{t} \rightarrow \tau, \bar{s} \rightarrow \sigma \text{ as } \varepsilon \rightarrow 0.$$

Set

$$\begin{aligned} \bar{\mu}_1 &:= u(\bar{y}, \bar{s}) - v(\bar{z}), \quad \bar{\mu}_2 := u(\bar{x}, \bar{t}) - v(\bar{z}) + \eta(\bar{t} - \bar{s}), \quad \bar{\mu} := \frac{\bar{\mu}_2}{\bar{\mu}_1}, \\ \bar{p} &:= \frac{\bar{y} - \bar{x}}{\varepsilon^2} \text{ and } \bar{q} := \frac{\bar{z} - \bar{x}}{\varepsilon^2}, \end{aligned}$$

and then we have

$$\bar{\mu}_1 \rightarrow \mu_1, \quad \bar{\mu}_2 \rightarrow \mu_2, \quad \bar{\mu} \rightarrow \mu \text{ as } \varepsilon \rightarrow 0.$$

Therefore we may assume that $\bar{\mu} < 1$ for small $\varepsilon > 0$.

Claim: There exists a constant $M_3 > 0$ such that

$$|\bar{p}| + |\bar{q}| \leq M_3$$

for all $\varepsilon, \delta, \alpha \in (0, 1)$.

We only consider the estimate of $|\bar{p}|$, since we can obtain the estimate of $|\bar{q}|$ similarly. The inequality $\Psi(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s}) \leq \Psi(\bar{x}, \bar{x}, \bar{z}, \bar{t}, \bar{s})$ implies

$$\begin{aligned} & \chi_1(\bar{x} - \bar{y}) \\ & \leq L_1 \left| \frac{1}{u(\bar{x}, \bar{s}) - v(\bar{z})} - \frac{1}{u(\bar{y}, \bar{s}) - v(\bar{z})} \right| + \alpha |\rho(\bar{x}) - \rho(\bar{y})| + |\phi(\bar{x}, \bar{s}) - \phi(\bar{y}, \bar{s})| \\ & \leq L_2 |\bar{x} - \bar{y}| \end{aligned} \tag{2.9}$$

for some $L_1, L_2 > 0$. Combining this (2.9) and the inequality in Lemma 4.3 (i) we get the conclusion of Claim for $M_3 := 4(M_1 + L_2)$.

In the sequel, we denote by $o_\varepsilon(1)$ a quantity which tends to 0 as $\varepsilon \rightarrow 0$. Derivating Ψ with respect to each variable t, s, x, y, z at $(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s})$ formally, we have

$$\begin{aligned} u_t(\bar{x}, \bar{t}) &= -\eta - 2\bar{\mu}_1(\bar{t} - \tau) = -\eta + o_\varepsilon(1), \\ u_s(\bar{y}, \bar{s}) &= -\frac{1}{\bar{\mu}}(\eta + \bar{\mu}_1\phi_s(\bar{y}, \bar{s})), \\ &= -\frac{1}{\mu}(\eta + \mu_1\phi_s(\xi, \sigma)) + o_\varepsilon(1), \\ D_x u(\bar{x}, \bar{t}) &= \bar{\mu}_1 \{ -D_x \chi_1(\bar{x} - \bar{y}) - D_x \chi_2(\bar{x} - \bar{z}) + 2(\xi - \bar{x}) - \alpha \tilde{n}(x) \} \\ &= \mu_1 \{ -D_x \chi_1(\bar{x} - \bar{y}) - D_x \chi_2(\bar{x} - \bar{z}) - \alpha \tilde{n}(\xi) \} + o_\varepsilon(1), \\ D_y u(\bar{y}, \bar{s}) &= \frac{\bar{\mu}_1}{\bar{\mu}} \{ D_y \chi_1(\bar{x} - \bar{y}) - D\phi(\bar{y}, \bar{s}) + \alpha \tilde{n}(y) \} \\ &= \frac{\mu_1}{\mu} \{ D_y \chi_1(\bar{x} - \bar{y}) - D\phi(\xi, \sigma) + \alpha \tilde{n}(\xi) \} + o_\varepsilon(1), \\ D_z v(\bar{z}) &= \frac{\bar{\mu}_1}{1 - \bar{\mu}} \{ D_z \chi_2(\bar{x} - \bar{z}) + \alpha \tilde{n}(\bar{z}) \} \\ &= \frac{\mu_1}{1 - \mu} \{ D_z \chi_2(\bar{x} - \bar{z}) + \alpha \tilde{n}(\xi) \} + o_\varepsilon(1). \end{aligned}$$

We remark that we should interpret $u_t, u_s, D_x u, D_y u$, and $D_z v$ as the viscosity solution sense here.

We first consider the case where $\bar{x} \in \partial\Omega$. In view of Claim, (A3)–(A5) and Lemma 4.3 (ii) we have

$$\begin{aligned}
& u_t(\bar{x}, \bar{t}) + B(\bar{x}, D_x u(\bar{x}, \bar{t})) \\
& \leq -\eta + B(\bar{x}, -\mu_1(D_x \chi_1(\bar{x} - \bar{y}) + D_x \chi_2(\bar{x} - \bar{z}))) - \theta\mu_1\alpha + o_\varepsilon(1) \\
& \leq \mu\left\{-\frac{\eta}{\mu} + B(\bar{x}, -\frac{\mu_1}{\mu}D_x \chi_1(\bar{x} - \bar{y}))\right\} + (1-\mu)B(\bar{x}, \frac{-\mu_1}{1-\mu}D_x \chi_2(\bar{x} - \bar{z}))\} \\
& \quad - \theta\mu_1\alpha + o_\varepsilon(1) \\
& \leq m(\delta + o_\varepsilon(1)) - \theta\mu_1\alpha + o_\varepsilon(1),
\end{aligned}$$

where m is a modulus. Therefore $u_t + B(\bar{x}, D_x u(\bar{x}, \bar{t})) < 0$ for $\varepsilon, \delta > 0$ which are small enough compared to $\alpha > 0$. Similarly if $\bar{z} \in \partial\Omega$ then we have $B(\bar{z}, D_z v(\bar{z})) > 0$.

We next consider the case where $\bar{y} \in \partial\Omega$. Note that we have

$$B(x, p+q) \geq B(x, p) + K(q \cdot n(x)) - M_B|q_T|,$$

for any $x \in \partial\Omega$, $p, q \in B(0, M_3)$, where $q_T := q - (q \cdot n(x))n(x)$ and K is the function defined by (2.5). By Lemma 4.3 and the homogeneity with degree 1 of F with respect to the p -variable, we have

$$\begin{aligned}
& u_s(\bar{y}, \bar{s}) + B(\bar{y}, D_y u(\bar{y}, \bar{s})) \\
& \geq -\frac{\mu_1}{\mu}\phi_s(\xi, \sigma) - \frac{\eta}{\mu} + B(\bar{y}, \frac{\mu_1}{\mu}D_y \chi_1(\bar{x} - \bar{y})) + \frac{\theta\mu_1\alpha}{\mu} \\
& \quad + K\left(-\frac{\mu_1}{\mu}D\phi(\xi, \sigma) \cdot n(\xi)\right) - M_B\left|-\frac{\mu_1}{\mu}D\phi(\xi, \sigma)\right|_T + o_\varepsilon(1) \\
& \geq -m(\delta + o_\varepsilon(1)) + \frac{\theta\mu_1\alpha}{\mu} - \frac{\mu_1}{\mu}(\phi_t(\xi, \sigma) + F(\xi, D\phi(\xi, \sigma))) + o_\varepsilon(1) \\
& \geq -m(\delta + o_\varepsilon(1)) + \frac{\theta\mu_1\alpha}{\mu} + o_\varepsilon(1).
\end{aligned}$$

Again, for $\varepsilon, \delta > 0$ which are small enough compared to $\alpha > 0$, we have

$$u_s(\bar{y}, \bar{s}) + B(\bar{y}, D_y u(\bar{y}, \bar{s})) > 0.$$

Therefore, by the definition of viscosity solutions we have

$$\begin{cases} -\eta + o_\varepsilon(1) + H(\bar{x}, D_x u(\bar{x}, \bar{t})) \geq 0, \\ -\frac{1}{\mu}(\eta + \mu_1\phi_s(\xi, \sigma)) + o_\varepsilon(1) + H(\bar{y}, D_y u(\bar{y}, \bar{s})) \leq 0, \\ H(\bar{z}, D_z v(\bar{z})) \leq 0. \end{cases} \quad (2.10)$$

In view of the above claim by taking a subsequence if necessary we may assume that

$$\begin{aligned}
& -\frac{\mu_1}{\mu}D_x \chi_1(\bar{x} - \bar{y}), \quad \frac{\mu_1}{\mu}D_y \chi_1(\bar{x} - \bar{y}) \rightarrow P, \text{ and} \\
& -\frac{\mu_1}{1-\mu}D_x \chi_2(\bar{x} - \bar{z}), \quad \frac{\mu_1}{1-\mu}D_z \chi_2(\bar{x} - \bar{z}) \rightarrow Q
\end{aligned}$$

as $\varepsilon \rightarrow 0$ for some $P, Q \in \mathbb{R}^n$.

Sending $\varepsilon \rightarrow 0$, $\delta \rightarrow 0$ and then $\alpha \rightarrow 0$ in (2.10), we obtain

$$\begin{cases} H(\xi, \mu P + (1 - \mu)Q) \geq \eta, \\ -\frac{1}{\mu}(\eta + \mu_1 \phi_s(\xi, \sigma)) + H(\xi, P - \frac{\mu_1}{\mu} D\phi(\xi, \sigma)) \leq 0, \\ H(\xi, Q) \leq 0. \end{cases}$$

We use these three inequality in the following way : first, using (A2), the second one leads to

$$-\frac{1}{\mu}(\eta + \mu_1 \phi_s(\xi, \sigma)) + H(\xi, P) - M \frac{\mu_1}{\mu} |D\phi(\xi, \sigma)| \leq 0,$$

for some constant $M > 0$. It remains to estimate $H(\xi, P)$.

Set $\tilde{P} := \mu(P - Q)$. By (A6)₊ we obtain

$$\begin{aligned} H(\xi, \mu P + (1 - \mu)Q) &= H(\xi, \tilde{P} + Q) \\ &\leq \mu H(\xi, \frac{\tilde{P}}{\mu} + Q) - \psi_\eta(1 - \mu) = \mu H(\xi, P) - \psi_\eta(1 - \mu) \end{aligned}$$

for some $\psi_\eta > 0$. We therefore have

$$\frac{1}{\mu}(\eta + \psi_\eta(1 - \mu)) \leq H(\xi, P).$$

Using this estimate in our first inequality yields the desired result, namely

$$\phi_t(\xi, \sigma) + M |D\phi(\xi, \sigma)| + \frac{\psi_\eta}{C}(\mu - 1) \geq 0. \quad \square$$

Remark 2.1. We remark that the solution of (CN) has the asymptotic monotonicity property. In order to prove this, we mainly use the following lemma in place of Lemma 2.2.

Lemma 2.3. *Set*

$$\begin{aligned} \mu_\eta^+(s) &:= \min_{x \in \bar{\Omega}, t \geq s} \left(\frac{u(x, t) - v(x) + \eta(t - s)}{u(x, s) - v(x)} \right), \\ \mu_\eta^-(s) &:= \max_{x \in \bar{\Omega}, t \geq s} \left(\frac{u(x, t) - v(x) - \eta(t - s)}{u(x, s) - v(x)} \right) \end{aligned}$$

for $\eta \in (0, \eta_0]$.

(i) Assume that (A6)₊ holds. The function μ_η^+ is a supersolution of

$$\max\{w(s) - 1, w'(s) + \frac{\psi_\eta}{C} \cdot (w(s) - 1)\} = 0 \text{ in } (0, \infty)$$

for any $\eta \in (0, \eta_0]$.

(ii) Assume that (A6)₋ holds. The function μ_η^- is a subsolution of

$$\min\{w(s) - 1, w'(s) + \frac{\psi_\eta}{C} \cdot \frac{w(s) - 1}{w(s)}\} = 0 \text{ in } (0, \infty)$$

for any $\eta \in (0, \eta_0]$.

3. ASYMPTOTIC BEHAVIOR II : THE OPTIMAL CONTROL/DYNAMICAL SYSTEM APPROACH

As we mentioned in the introduction, we mainly concentrate on Problem (CN) in this section.

3.1. Variational formulas for (CN) and (DBC). We begin this section with an introduction to the Skorokhod problem. Let $x \in \partial\Omega$ and set

$$G(x, \xi) = \sup_{p \in \mathbb{R}^n} (\xi \cdot p - B(x, p)) \quad \text{for } \xi \in \mathbb{R}^n,$$

$$\mathcal{G}(x) = \{(\gamma, g) \in \mathbb{R}^n \times \mathbb{R} : B(x, p) \geq \gamma \cdot p - g \quad \text{for all } p \in \mathbb{R}^n\}.$$

By the convex duality, we have

$$\mathcal{G}(x) = \{(\gamma, g) \in \mathbb{R}^{n+1} : g \geq G(x, \gamma)\}.$$

Note that

$$\begin{aligned} G(x, \xi) &\geq -B(x, 0) \geq -\max_{y \in \partial\Omega} B(y, 0), \\ \bigcup_{p \in \mathbb{R}^n} \partial_p B(x, p) &\subset \{\gamma \in \mathbb{R}^n : G(x, \gamma) < \infty\}, \\ \overline{\bigcup_{p \in \mathbb{R}^n} \partial_p B(x, p)} &= \overline{\{\gamma \in \mathbb{R}^n : G(x, \gamma) < \infty\}}. \end{aligned}$$

We set

$$\Gamma(x) = \overline{\bigcup_{p \in \mathbb{R}^n} \partial_p B(x, p)},$$

and observe that $\Gamma(x) \subset \bar{B}_M$, $\gamma \cdot \tilde{n}(x) \geq \theta$ for $\gamma \in \Gamma(x)$, $\mathcal{G}(x)$ is a convex subset of \mathbb{R}^{n+1} and $\Gamma(x)$ is a closed convex subset of \mathbb{R}^n . Observe as well that if $(\gamma, g) \in \mathbb{R}^{n+1}$ belongs to $\mathcal{G}(x)$ for some $x \in \partial\Omega$, then $G(x, \gamma) < \infty$ and hence $\gamma \in \Gamma(x)$.

For example, if $B(x, p) = \gamma(x) \cdot p - g(x)$ for some functions $\gamma, g \in C(\partial\Omega)$, then

$$G(x, \xi) = \begin{cases} g(x) & \text{if } \xi = \gamma(x), \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\mathcal{G}(x) = \{(\gamma(x), r) : r \geq g(x)\} = \{\gamma(x)\} \times [g(x), \infty), \quad \Gamma(x) = \{\gamma(x)\}.$$

Let $x \in \bar{\Omega}$ and $0 < T < \infty$, and let $\eta \in \text{AC}([0, T], \mathbb{R}^n)$, $v \in L^1([0, T], \mathbb{R}^n)$ and $l \in L^1([0, T], \mathbb{R})$. We introduce a set of conditions:

$$\begin{cases} \eta(0) = x, \\ \eta(t) \in \bar{\Omega} \quad \text{for all } t \in [0, T], \\ l(t) \geq 0 \quad \text{for a.e. } t \in [0, T], \\ l(t) = 0 \quad \text{if } \eta(t) \in \Omega \quad \text{for a.e. } t \in [0, T], \end{cases} \quad (3.1)$$

and

$$\text{there exists a function } f \in L^1([0, T], \mathbb{R}) \text{ such that} \quad (3.2)$$

$$((v - \dot{\eta})(t), f(t)) \in l(t)\mathcal{G}(\eta(t)) \quad \text{for a.e. } t \in [0, T].$$

Observe here that the inclusion

$$((v - \dot{\eta})(t), f(t)) \in l(t)\mathcal{G}(\eta(t))$$

is equivalent to the condition that $f(t) \geq l(t)G(\eta(t), l(t)^{-1}(v - \dot{\eta})(t))$ if $l(t) > 0$, and $\dot{\eta}(t) = v(t)$ and $f(t) = 0$ if $l(t) = 0$. Condition (3.2) is therefore equivalent to the condition that

$$t \mapsto l(t)G(\eta(t), l(t)^{-1}(v - \dot{\eta})(t)) \quad \text{is integrable on } [0, T], \quad (3.3)$$

$$\text{and } \dot{\eta}(t) = v(t) \text{ if } l(t) = 0 \text{ for a.e. } t \in [0, T].$$

Here we have used the fact that G is lower semi-continuous (hence Borel) function bounded from below by the constant $-\max_{x \in \partial\Omega} B(x, 0)$. The expression $l(t)G(\eta(t), l(t)^{-1}(v - \dot{\eta})(t))$ in (3.3) is actually defined only for those $t \in [0, T]$ such that $l(t) > 0$, but we understand that

$$l(t)G(\eta(t), l(t)^{-1}(v - \dot{\eta})(t)) = \begin{cases} l(t)G(\eta(t), l(t)^{-1}(v - \dot{\eta})(t)) & \text{if } l(t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly we henceforth use the convention that zero times an undefined quantity equals zero. With use of this convention, we define the function $F(\eta, v, l)$ on $[0, T]$ by

$$F(\eta, v, l)(t) = l(t)G(\eta(t), l(t)^{-1}(v(t) - \dot{\eta}(t))).$$

We remark that under assumption (3.3) we have for a.e. $t \in [0, T]$,

$$F(\eta, v, l)(t) \geq (v - \dot{\eta})(t) \cdot p - l(t)B(\eta(t), p) \quad \text{for all } p \in \mathbb{R}^n. \quad (3.4)$$

In the case where $B(x, p) = \gamma(x) \cdot p - g(x)$ for some $\gamma, g \in C(\partial\Omega)$, it is easily seen that condition (3.3) is equivalent to

$$\dot{\eta}(t) + l(t)\gamma(\eta(t)) = v(t) \quad \text{for a.e. } t \in [0, T].$$

(Compare this together with (3.1) with (1.4) in [22].) In this case we have $F(\eta, v, l)(t) = l(t)g(\eta(t))$ for a.e. $t \in [0, T]$.

Now, given a point $x \in \bar{\Omega}$, a constant $0 < T < \infty$ and a function $v \in L^1([0, T], \mathbb{R}^n)$, the *Skorokhod problem* is to find a pair of functions $\eta \in \text{AC}([0, T], \mathbb{R}^n)$ and $l \in L^1([0, T], \mathbb{R})$ for which (3.1) and (3.2) are satisfied.

Theorem 3.1. *Let $x \in \bar{\Omega}$, $0 < T < \infty$ and $v \in L^1([0, T], \mathbb{R}^n)$. There exists a solution (η, l) of the Skorokhod problem. Moreover, there exists a constant $C > 0$, independent of x , T and v , such that, for any solution (η, l) of the Skorokhod problem, the inequalities $|\dot{\eta}(t)| \leq C|v(t)|$ and $l(t) \leq C|v(t)|$ hold for a.e. $t \in [0, T]$.*

For fixed $x \in \bar{\Omega}$ and $0 < T < \infty$, $\text{SP}_T(x)$ denotes the set of the all triples (η, v, l) of functions $\eta \in \text{AC}([0, T], \mathbb{R}^n)$, $v \in L^1([0, T], \mathbb{R}^n)$ and $l \in L^1([0, T], \mathbb{R})$ such that conditions (3.1), (3.2) are satisfied, and $\text{SP}(x)$ denotes the set of the triples (η, v, l) of functions η, v and l on $[0, \infty)$ such that, for all $0 < T < \infty$, the restriction of (η, v, l) to the interval $[0, T]$ belongs to $\text{SP}_T(x)$.

Note here that if $(\eta, v, l) \in \text{SP}_T(x)$ for some $x \in \bar{\Omega}$ and $0 < T < \infty$ and if we extend the domain of (η, v, l) to $[0, \infty)$ by setting

$$\eta(t) = \eta(T), \quad v(t) = 0, \quad \text{and} \quad l(t) = 0 \quad \text{for } t > T,$$

then the extended (η, v, l) belongs to $\text{SP}(x)$.

Moreover we set

$$\text{SP}_T = \bigcup_{x \in \bar{\Omega}} \text{SP}_T(x) \quad \text{and} \quad \text{SP} = \bigcup_{x \in \bar{\Omega}} \text{SP}(x).$$

Let $u_0 \in C(\bar{\Omega})$. For $t > 0$ we set

$$U(x, t) = \inf \left\{ \int_0^t (L(\eta(s), -v(s)) + f(s)) \, ds + u_0(\eta(t)) : (\eta, v, l) \in \text{SP}(x) \right\}, \quad (3.5)$$

where $L(x, \xi) := \sup_{p \in \mathbb{R}^n} (\xi \cdot p - H(x, p))$ and $f := F(\eta, v, l)$. We call the function L the Hamiltonian of H . This function has the properties: $L(x, \xi)$ is lower semicontinuous on $\bar{\Omega} \times \mathbb{R}^n$, convex in $\xi \in \mathbb{R}^n$ and coercive, i.e., $\lim_{|\xi| \rightarrow \infty} L(x, \xi) = \infty$ for all $x \in \bar{\Omega}$. The function L may take the value ∞ , but $\sup_{\bar{\Omega} \times B_r} L < \infty$ for some constant $r > 0$.

Theorem 3.2. *The function U is continuous on $\bar{\Omega} \times (0, \infty)$ and a solution of*

$$u_t + H(x, Du) = 0 \quad \text{in } \Omega \times (0, \infty), \quad (3.6)$$

$$B(x, Du) = 0 \quad \text{on } \partial\Omega \times (0, \infty). \quad (3.7)$$

Moreover, we have

$$\lim_{t \rightarrow 0+} U(x, t) = u_0(x) \quad \text{uniformly on } \bar{\Omega}.$$

We set $U(x, 0) = u_0(x)$ for $x \in \bar{\Omega}$. The above theorem ensures that $U \in C(\bar{\Omega} \times [0, \infty))$ and U is a solution of (CN).

In what follows we give an outline of proofs of Theorems 3.1 and 3.2. Indeed, most of the arguments parallel to those for similar assertions in [22] for (CN) with linear Neumann boundary condition.

Lemma 3.3. *Let $\psi \in C(\partial\Omega, \mathbb{R}^n)$ and $\varepsilon > 0$. There exists $(\gamma, g) \in C(\partial\Omega, \mathbb{R}^{n+1})$ such that for all $x \in \partial\Omega$, $(\gamma(x), g(x)) \in \mathcal{G}(x)$ and $B(x, \psi(x)) < \varepsilon + \gamma(x) \cdot \psi(x) - g(x)$.*

Proof. Let $\delta > 0$ and set

$$B_\delta(x, p) = \inf_{q \in \mathbb{R}^n} \left(B(x, q) + \frac{1}{2\delta} |p - q|^2 \right) \quad \text{for all } (x, p) \in \partial\Omega \times \mathbb{R}^n.$$

Note that $B_\delta \leq B$ on $\partial\Omega \times \mathbb{R}^n$, that, as $\delta \rightarrow 0$, $B_\delta(x, p) \rightarrow B(x, p)$ uniformly on $\partial\Omega \times B_R(0)$ for every $R > 0$ and that for $x \in \partial\Omega$, the function $p \mapsto B_\delta(x, p)$ is in $C^{1+1}(\mathbb{R}^n)$. Also, by (A4) we see that

$$B_\delta(x, p) = \min_{|p-q| \leq R} \left(B(x, q) + \frac{1}{2\delta} |p - q|^2 \right) \quad \text{for } (x, p) \in \partial\Omega \times \mathbb{R}^n$$

for some $R > 0$ depending only on δ and M_B . Hence, $B_\delta \in C(\partial\Omega \times \mathbb{R}^n)$. Moreover, it is easy to see that $D_p B_\delta \in C(\partial\Omega \times \mathbb{R}^n)$. Indeed, if $(x_j, p_j) \rightarrow (y, q)$ as $j \rightarrow \infty$ and $\xi_j := D_p B_\delta(x_j, p_j)$, then

$$B_\delta(x_j, p) \geq B_\delta(x_j, p_j) + \xi_j \cdot (p - p_j) \quad \text{for all } p \in \mathbb{R}^n.$$

Noting that $|\xi_j| \leq M_B$, we may choose a subsequence $\{\xi_{j_k}\}_{k \in \mathbb{N}}$, converging to a point η , of $\{\xi_j\}$. From the above inequality with $j = j_k$, we get in the limit

$$B_\delta(y, p) \geq \eta \cdot (p - q) + B_\delta(y, q) \quad \text{for } p \in \mathbb{R}^n.$$

This shows that $\eta = D_p B_\delta(y, q)$, which implies that $\lim_{j \rightarrow \infty} D_p B_\delta(x_j, p_j) = D_p B_\delta(y, q)$ and $D_p B_\delta \in C(\partial\Omega \times \mathbb{R}^n)$.

If we set $\gamma(x) = D_p B_\delta(x, \psi(x))$ and $g(x) = \gamma(x) \cdot \psi(x) - B_\delta(x, \psi(x))$, then we have for all $(x, p) \in \partial\Omega \times \mathbb{R}^n$,

$$B(x, p) \geq B_\delta(x, p) \geq \gamma(x) \cdot (p - \psi(x)) + B_\delta(x, \psi(x)) = \gamma(x) \cdot p - g(x).$$

Thus, we find that $(\gamma(x), g(x)) \in \mathcal{G}(x)$ for all $x \in \partial\Omega$. Moreover, for each fixed $\varepsilon > 0$, if $\delta > 0$ is small enough, then we have

$$B(x, \psi(x)) < \varepsilon + B_\delta(x, \psi(x)) = \varepsilon + \gamma(x) \cdot \psi(x) - g(x).$$

Finally, we note that $(\gamma, g) \in C(\partial\Omega, \mathbb{R}^{n+1})$ and conclude the proof. \square

Lemma 3.4. *Let $0 < T < \infty$. There is a constant $C > 0$, depending only on θ and M_B , such that for any $(\eta, v, l) \in \text{SP}_T$,*

$$\max\{|\dot{\eta}(s)|, l(s)\} \leq C|v(s)| \quad \text{for a.e. } s \in [0, T].$$

Recall that $M_B > 0$ is a Lipschitz bound of the functions $p \mapsto B(x, p)$ for all $x \in \partial\Omega$.

An immediate consequence of the above lemma is that for $(\eta, v, l) \in \text{SP}$, if I is an interval of $[0, \infty)$ and $v \in L^p(I, \mathbb{R}^n)$, with $1 \leq p \leq \infty$, then $(\dot{\eta}, l) \in L^p(I, \mathbb{R}^{n+1})$.

Proof. Let $(\eta, v, l) \in \text{SP}_T$ and set $\xi = v - \dot{\eta}$ on $[0, \infty)$. We choose a function $f \in L^1([0, T], \mathbb{R})$ so that $((v - \dot{\eta})(s), f(s)) \in l(s)\mathcal{G}(\eta(s))$ for a.e. $s \in [0, T]$. If $l(s) = 0$ for a.e. $s \in [0, T]$, then we have $\dot{\eta}(s) = v(s)$ for a.e. $s \in [0, T]$, which yields $\max\{|\dot{\eta}(s)|, l(s)\} = |v(s)|$ for a.e. $s \in [0, T]$, and we are done. Henceforth we assume that the set $E := \{s \in [0, T] : l(s) > 0\}$ has positive measure. We choose a subset E_0 of E having full measure so that $E_0 \subset (0, T)$, that $\eta(s) \in \partial\Omega$ and $((v - \dot{\eta})(s), f(s)) \in l(s)\mathcal{G}(\eta(s))$ for all $s \in E_0$ and that η is differentiable everywhere in E_0 . We set $\gamma(s) = l(s)^{-1}(v(s) - \dot{\eta}(s))$ for $s \in E_0$, and note that $\gamma(s) \in \Gamma(\eta(s))$ for all $s \in E_0$.

Using the defining function ρ (cf. (A0)) and noting that $\rho(\eta(s)) \leq 0$ for all $s \in [0, T]$, we find that for any $s \in E_0$,

$$0 = \frac{d}{ds} \rho(\eta(s)) = D\rho(\eta(s)) \cdot \dot{\eta}(s) = |D\rho(\eta(s))| n(\eta(s)) \cdot (v(s) - l(s)\gamma(s)).$$

That is, $n(\eta(s)) \cdot v(s) = l(s)n(\eta(s)) \cdot \gamma(s)$ for all $s \in E_0$. Fix any $s \in E_0$. Since $\gamma(s) \in \Gamma(\eta(s))$, we have $|\gamma(s)| \leq M$ and $\gamma(s) \cdot n(\eta(s)) \geq \theta|\gamma(s)|$. Accordingly, we get

$$|v(s)| \geq n(\eta(s)) \cdot v(s) = n(\eta(s)) \cdot l(s)\gamma(s) \geq l(s)\theta,$$

and hence, $l(s) \leq |v(s)|/\theta$. Finally, we note that $|\dot{\eta}(s)| \leq |v(s)| + |\xi(s)| \leq (1 + M/\theta)|v(s)|$, which completes the proof. \square

Proof of Theorem 3.1. Fix $x \in \bar{\Omega}$, $0 < T < \infty$ and $v \in L^1([0, T], \mathbb{R}^n)$. Due to Lemma 3.3, there exists a $(\gamma, g) \in C(\partial\Omega, \mathbb{R}^{n+1})$ such that $(\gamma(x), g(x)) \in \mathcal{G}(x)$ for all $x \in \partial\Omega$. According to [22, Theorem 4.1], there exists a pair $(\eta, l) \in AC([0, T], \mathbb{R}^n) \times L^1([0, T], \mathbb{R})$ such that $\eta(0) = x$, $\eta(s) \in \bar{\Omega}$ for all $s \in [0, T]$ and for a.e. $s \in [0, T]$,

$$l(s) \geq 0, \quad l(s) = 0 \quad \text{if } \eta(s) \in \Omega, \quad \text{and} \quad \dot{\eta}(s) + l(s)\gamma(\eta(s)) = v(s).$$

We set $f(s) = l(s)g(\eta(s))$ for $s \in [0, T]$, and observe that we have for a.e. $s \in [0, T]$,

$$((v - \dot{\eta})(s), f(s)) = l(s)(\gamma(s), g(s)) \in l(s)\mathcal{G}(\eta(s)),$$

completing the existence part of the proof. The remaining part of the proof is exactly what Lemma 3.4 guarantees. \square

Proof of Theorem 3.2. Set $Q = \bar{\Omega} \times (0, \infty)$. We first prove that U is a sub-solution of (3.6), (3.7). Let $(\hat{x}, \hat{t}) \in Q$ and $\phi \in C^1(Q)$. Assume that $U^* - \phi$ attains a strict maximum at (\hat{x}, \hat{t}) . We need to show that if $\hat{x} \in \Omega$, then $\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D\phi(\hat{x}, \hat{t})) \leq 0$, and if $\hat{x} \in \partial\Omega$, then either

$$\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D\phi(\hat{x}, \hat{t})) \leq 0 \quad \text{or} \quad B(\hat{x}, D\phi(\hat{x}, \hat{t})) \leq 0. \quad (3.8)$$

We are here concerned only with the case where $\hat{x} \in \partial\Omega$. The other case can be treated similarly. To prove (3.8), we argue by contradiction. Thus we suppose that (3.8) were false. We may choose an $\varepsilon \in (0, 1)$ so that $\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D\phi(\hat{x}, \hat{t})) > \varepsilon$ and $B(\hat{x}, D\phi(\hat{x}, \hat{t})) > \varepsilon$.

By Lemma 3.3, we may choose $(\gamma, g) \in C(\partial\Omega, \mathbb{R}^{n+1})$ so that $(\gamma(x), g(x)) \in \mathcal{G}(x)$ for all $x \in \partial\Omega$ and $B(\hat{x}, D\phi(\hat{x}, \hat{t})) < \varepsilon + \gamma(\hat{x}) \cdot D\phi(\hat{x}, \hat{t}) - g(\hat{x})$. Note that $\gamma(\hat{x}) \cdot D\phi(\hat{x}, \hat{t}) - g(\hat{x}) > 0$. Set $R = \bar{B}_{2\varepsilon}(\hat{x}) \times [\hat{t} - 2\varepsilon, \hat{t} + 2\varepsilon]$. By replacing $\varepsilon > 0$ if needed, we may assume that $\hat{t} - 2\varepsilon > 0$ and for all $(x, t) \in R \cap Q$,

$$\phi_t(x, t) + H(x, D\phi(x, t)) \geq \varepsilon \quad \text{and} \quad \gamma(x) \cdot D\phi(x, t) - g(x) \geq 0, \quad (3.9)$$

where γ and g are assumed to be defined and continuous on $\bar{\Omega}$. We may assume that $(U^* - \phi)(\hat{x}, \hat{t}) = 0$. Set $m = -\max_{Q \cap \partial R}(U^* - \phi)$, and note that $m > 0$ and $U(x, t) \leq \phi(x, t) - m$ for $(x, t) \in Q \cap \partial R$. We choose a point $(\bar{x}, \bar{t}) \in (\bar{B}_\varepsilon(\hat{x}) \times [\hat{t} - \varepsilon, \hat{t} + \varepsilon]) \cap Q$ so that $(U - \phi)(\bar{x}, \bar{t}) > -m$.

Now, we consider the Skorokhod problem with the function $\gamma(x) \cdot p - g(x)$ in place of $B(x, p)$. For the moment we denote by $\text{SP}_T(x; \gamma, g)$ the set of all $(\eta, v, l) \in AC([0, T], \mathbb{R}^n) \times L^1([0, T], \mathbb{R}^n) \times L^1([0, T], \mathbb{R})$ satisfying (3.1) and

(3.2), with the function $\gamma(x) \cdot p - g(x)$ in place of $B(x, p)$. We apply [22, Lemma 5.5], to find a triple $(\eta, v, l) \in \text{SP}_{\bar{t}}(\bar{x}; \gamma, g)$ such that for a.e. $s \in (0, \bar{t})$,

$$H(\eta(s), D\phi(\eta(s), \bar{t} - s)) + L(\eta(s), -v(s)) \leq \varepsilon - v(s) \cdot D\phi(\eta(s), \bar{t} - s) \quad (3.10)$$

Note here that, since $(\eta, v, l) \in \text{SP}_{\sigma}(\bar{x}; \gamma, g)$, we have $\dot{\eta}(t) + l(s)\gamma(s) = v(s)$ and $F(\eta, v, l)(s) = l(s)g(\eta(s))$ for a.e. $s \in [0, \bar{t}]$.

We set $\sigma = \min\{s \geq 0 : (\eta(s), \bar{t} - s) \in \partial R\}$ and note that $(\eta(s), \bar{t} - s) \in Q \cap R$ for all $0 \leq s \leq \sigma$ and $0 < \sigma \leq \bar{t}$. Using the dynamic programming principle, we obtain

$$\begin{aligned} \phi(\bar{x}, \bar{t}) &< U(\bar{x}, \bar{t}) + m \\ &\leq \int_0^\sigma (L(\eta(s), -v(s)) + g(\eta(s))l(s)) \, ds + U(\eta(\sigma), \bar{t} - \sigma) + m \\ &\leq \int_0^\sigma (L(\eta(s), -v(s)) + g(\eta(s))l(s)) \, ds + \phi(\eta(\sigma), \bar{t} - \sigma). \end{aligned}$$

Hence, setting $p(s) := D\phi(\eta(s), \bar{t} - s)$, we get

$$\begin{aligned} 0 &< \int_0^\sigma (L(\eta(s), -v(s)) + g(\eta(s))l(s) + \frac{d}{ds}\phi(\eta(s), \bar{t} - s)) \, ds \\ &\leq \int_0^\sigma (L(\eta(s), -v(s)) + g(\eta(s))l(s) + p(s) \cdot (v(s) - l(s)\gamma(\eta(s))) \\ &\quad - \phi_t(\eta(s), \bar{t} - s)) \, ds. \end{aligned}$$

Using (3.10) and (3.9), we obtain

$$0 < \int_0^\sigma \{ \varepsilon - H(\eta(s), p(s)) - \phi_t(\eta(s), \bar{t} - s) + l(s)(g(\eta(s)) - \gamma(\eta(s)) \cdot p(s)) \} \, ds \leq 0,$$

which is a contradiction. Thus, U is a subsolution of (3.6), (3.7).

Next, we turn to the proof of the supersolution property of U . Let $\phi \in C^1(Q)$ and $(\hat{x}, \hat{t}) \in Q$. Assume that $U_* - \phi$ attains a strict minimum at (\hat{x}, \hat{t}) . We need to show that if $\hat{x} \in \Omega$, then $\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D\phi(\hat{x}, \hat{t})) \geq 0$, and if $\hat{x} \in \partial\Omega$, then either

$$\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D\phi(\hat{x}, \hat{t})) \geq 0 \quad \text{or} \quad B(\hat{x}, D\phi(\hat{x}, \hat{t})) \geq 0. \quad (3.11)$$

As before, we only consider the case where $\hat{x} \in \partial\Omega$. To prove (3.11), we suppose by contradiction that $\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D\phi(\hat{x}, \hat{t})) < 0$ and $B(\hat{x}, D\phi(\hat{x}, \hat{t})) < 0$. There is a constant $\varepsilon > 0$ such that

$$\begin{aligned} \phi_t(x, t) + H(x, D\phi(x, t)) &< 0 \quad \text{and} \\ B(x, D\phi(x, t)) &< 0 \quad \text{for all } (x, t) \in R \cap Q, \end{aligned} \quad (3.12)$$

where $R := \bar{B}_{2\varepsilon}(\hat{x}) \times [\hat{t} - 2\varepsilon, \hat{t} + 2\varepsilon]$. Here we may assume that $\hat{t} - 2\varepsilon > 0$ and $(U_* - \phi)(\hat{x}, \hat{t}) = 0$.

Set $m := \min_{Q \cap \partial R} (U_* - \phi) (> 0)$. We may choose a point $(\bar{x}, \bar{t}) \in (B_\varepsilon(\hat{x}) \times (\hat{t} - \varepsilon, \hat{t} + \varepsilon)) \cap Q$ so that $(U - \phi)(\bar{x}, \bar{t}) < m$. We select a triple $(\eta, v, l) \in \text{SP}(\bar{x})$

so that

$$U(\bar{x}, \bar{t}) + m > \int_0^{\bar{t}} (L(\eta(s), -v(s)) + f(s)) \, ds + u_0(\eta(\bar{t})),$$

where $f := F(\eta, v, l)$. We set $\sigma = \min\{s \geq 0 : (\eta(s), \bar{t} - s) \in \partial R\}$. It is clear that $\sigma > 0$ and $\eta(s) \in R \cap Q$ for all $s \in [0, \sigma]$. Accordingly, we have

$$\begin{aligned} \phi(\bar{x}, \bar{t}) + m &> \int_0^\sigma (L(\eta(s), -v(s)) + f(\eta(s))) \, ds + U(\eta(\sigma), \bar{t} - \sigma) \\ &\geq \int_0^\sigma (L(\eta(s), -v(s)) + f(\eta(s))) \, ds + \phi(\eta(\sigma), \bar{t} - \sigma) + m, \end{aligned}$$

and hence,

$$0 > \int_0^\sigma (L(\eta(s), -v(s)) + f(\eta(s)) + D\phi(\eta(s), \bar{t} - s) \cdot \dot{\eta}(s) - \phi_t(\eta(s), \bar{t} - s)) \, ds.$$

Note by the Fenchel-Young inequality and (3.4) that for a.e. $s \in [0, \sigma]$,

$$L(\eta(s), -v(s)) + f(s) \geq -\dot{\eta}(s) \cdot p(s) - H(\eta(s), p(s)) - l(s)B(\eta(s), p(s)),$$

where $p(s) := D\phi(\eta(s), \bar{t} - s)$. Consequently, in view of (3.12) we get

$$0 > \int_0^\sigma (-H(\eta(s), p(s)) - \phi_t(\eta(s), \bar{t} - s) - l(s)B(\eta(s), p(s))) \, ds \geq 0,$$

which is a contradiction. The function U is thus a supersolution of (3.6), (3.7).

It remains to show the continuity of U on $\bar{\Omega} \times [0, \infty)$. In view of Theorem 1.1, we need only to prove that

$$U^*(x, 0) \leq U_*(x, 0) \quad \text{for all } x \in \bar{\Omega}. \quad (3.13)$$

Indeed, once this is done, we see by Theorem 1.1 that $U^* \leq U_*$ on $\bar{\Omega} \times [0, \infty)$, which guarantees that $U \in C(\bar{\Omega} \times [0, \infty))$.

To show (3.13), fix any $\varepsilon > 0$. We may select a function $u_0^\varepsilon \in C^1(\bar{\Omega})$ such that $B(x, Du_0^\varepsilon(x)) \leq 0$ for all $x \in \partial\Omega$ and

$$|u_0(x) - u_0^\varepsilon(x)| \leq \varepsilon \quad \text{for all } x \in \bar{\Omega}.$$

Indeed, we can first approximate u_0 by a sequence of C^1 functions and then modify the normal derivative (without modifying too much the function itself) by adding a function of the form $\varepsilon\zeta(C\rho(x)/\varepsilon)$ where ζ is a C^1 , increasing function such that $\zeta(0) = 0$, $\zeta'(0) = 1$ and $-1 \leq \zeta(r) \leq 1$ for all $r \in \mathbb{R}$.

Then we may choose a constant $C_\varepsilon > 0$ so that the function $\psi(x, t) := u_0^\varepsilon(x) - C_\varepsilon t$ is a (classical) subsolution of (3.6), (3.7). Then, for any $(x, t) \in Q$ and $(\eta, v, l) \in \text{SP}(x)$, we have

$$\psi(\eta(t), 0) - \psi(\eta(0), t) = \int_0^t (D\psi(\eta(s), t - s) \cdot \dot{\eta}(s) - \psi_t(\eta(s), t - s)) \, ds.$$

Setting $p(s) = D\psi(\eta(s), t - s)$ and $f(s) = F(\eta, v, l)(s)$ for $s \in [0, t]$ and using the Fenchel-Young inequality, we observe that for a.e. $s \in [0, t]$,

$$p(s) \cdot \dot{\eta}(s) - \psi_t(\eta(s), t - s) \geq -L(\eta(s), -v(s)) - f(s).$$

Combining these observations, we obtain

$$\psi(x, t) \leq \int_0^t (L(\eta(s), -v(s)) + f(s)) ds + u_0^\varepsilon(x),$$

which ensures that $U(x, t) \geq u_0(x) - 2\varepsilon - C_\varepsilon t$ for all $(x, t) \in \bar{\Omega} \times [0, \infty)$, and moreover, $U_*(x, 0) \geq u_0(x)$ for all $x \in \bar{\Omega}$.

Next, fix any $(x, t) \in Q$ and set $\eta(s) = x$, $v(s) = 0$ and $l(s) = 0$ for $s \geq 0$. Observe that $(\eta, v, l) \in \text{SP}(x)$ and that $F(\eta, v, l) = 0$ and

$$U(x, t) \leq \int_0^t L(x, 0) ds + u_0(x) = L(x, 0)t + u_0(x) \leq u_0(x) - t \min_{x \in \bar{\Omega}} H(x, 0).$$

This shows that $U^*(x, 0) \leq u_0(x)$ for all $x \in \bar{\Omega}$. Thus we find that (3.13) is valid, which completes the proof. \square

Next we present the variational formula for the solution of (DBC). The basic idea of obtaining this formula is similar to that for (CN), and thus we just outline it or skip the details.

We define the function W on $Q := \bar{\Omega} \times (0, \infty)$ by

$$W(x, t) = \inf \left\{ \int_0^\sigma (L(\eta(s), -v(s)) + f(s)) ds + u_0(\eta(\sigma)) \right\}, \quad (3.14)$$

where the infimum is taken all over $(\eta, v, l) \in \text{SP}(x)$, $f = F(\eta, v, l)$, and $\sigma \in (0, t]$ is given by $t = \int_0^\sigma (1 + l(r)) dr$. Then we extend the domain of definition of W to \bar{Q} by setting $W(x, 0) = u_0(x)$ for $x \in \bar{\Omega}$.

In the definition of W we apparently use the set SP (the Skorokhod problem for Ω and B), but the underlining idea is to consider the Skorokhod problem for the domain $\Omega \times \mathbb{R}$ and the function $B(x, p) + q$ in place of Ω and $B(x, p)$, respectively. Indeed, setting $\hat{\Omega} = \Omega \times \mathbb{R}$, $\hat{B}(x, p, q) = B(x, p) + q$ and $\hat{H}(x, p, q) = H(x, p) + q$, we observe that the vector $(\tilde{n}(x), 0)$ is the unit outer normal at $(x, t) \in \partial\hat{\Omega}$, the conditions (A1)–(A7) are satisfied with \hat{B} and $\hat{\Omega}$ in place of B and Ω and the Lagrangian \hat{L} of \hat{H} is given by

$$\hat{L}(x, \xi, \eta) = \sup_{(p, q) \in \mathbb{R}^{n+1}} (p \cdot \xi + q\eta - \hat{H}(x, p)) = L(x, \xi) + \delta_{\{1\}}(\eta), \quad (3.15)$$

where $\delta_{\{1\}}$ is the indicator function of the set $\{1\}$, i.e., $\delta_{\{1\}}(\eta) = 0$ if $\eta = 1$ and $= \infty$ if $\eta \neq 1$. If we set for $(x, t) \in \partial\hat{\Omega}$,

$\hat{\mathcal{G}}(x, t) = \{(\gamma, \delta, g) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \hat{B}(x, p, q) \geq \gamma \cdot p + \delta q - g \text{ for } (p, q) \in \mathbb{R}^{n+1}\}$, then it is easily seen that $\hat{\mathcal{G}}(x, t) = \{(\gamma, 1, g) : (\gamma, g) \in \mathcal{G}(x)\}$.

The Skorokhod problem for $\hat{\Omega}$ and \hat{B} is to find for given $(x, t) \in \bar{\hat{\Omega}}$, $T > 0$ and $(v, w) \in L^1([0, T], \mathbb{R}^{n+1})$ a pair of functions $(\eta, \tau) \in \text{AC}([0, T], \mathbb{R}^{n+1})$ and $l \in L^1([0, T], \mathbb{R})$ such that $(\eta(0), \tau(0)) = (x, t)$, $(\eta(s), \tau(s)) \in \hat{\Omega}$ for $s \in [0, T]$, $l(s) \geq 0$ for a.e. $s \in [0, T]$, $l(s) = 0$ if $(\eta(s), \tau(s)) \in \hat{\Omega}$ for a.e. $s \in [0, T]$, and $((v - \dot{\eta})(s), (w - \dot{\tau})(s), f(s)) \in l(s)\hat{\mathcal{G}}(\eta(s), \tau(s))$ for a.e. $s \in [0, T]$ and for some $f \in L^1([0, T], \mathbb{R})$. It is easily checked that for given $(x, t) \in \bar{\hat{\Omega}}$, $T > 0$ and $(v, w) \in L^1([0, T], \mathbb{R}^{n+1})$, the pair of functions $(\eta, \tau) \in \text{AC}([0, T], \mathbb{R}^{n+1})$ and

$l \in L^1([0, T], \mathbb{R})$ is a solution of the Skorokhod problem for $\hat{\Omega}$ and \hat{B} if and only if $(\eta, v, l) \in \text{SP}_T(x)$ and $\tau(s) = t - \int_0^s (w(r) + l(r)) dr$ for all $s \in [0, T]$. If we take into account of the form (3.15), then we need to consider the Skorokhod problem only with $w(s) = 1$. That is, in our minimization at $(x, t) \in Q$, we have only to consider the infimum all over $(\eta, v, l) \in \text{SP}(x)$ and τ such that $\tau(s) = t - \int_0^s (1 + l(r)) dr$ for $s \geq 0$. Note that this function τ is decreasing on $[0, \infty)$ and that $\tau(s) = 0$ if and only if $t = \int_0^s (1 + l(r)) dr$, which justifies the choice of σ in (3.14).

We have the following theorems concerning (DBC).

Theorem 3.5. *The function W is a solution of (DBC) and continuous on \bar{Q} . Moreover, if $u_0 \in \text{Lip}(\bar{\Omega})$, then $W \in \text{Lip}(\bar{Q})$.*

In the above theorem, the subsolution (resp., supersolution) property of (DBC) assumes as well the inequality $u(\cdot, 0) \leq u_0$ (resp., $v(\cdot, 0) \geq u_0$) on $\partial\Omega$.

We do not give here the proof of the above theorem, since one can easily adapt the proof of Theorems 3.2, using theorems 1.1 and 1.2, with minor modifications. A typical modification is the following: in the proof of the viscosity property of W , we have to replace the curves $(\eta(s), \bar{t} - s)$, with $s \geq 0$, which are used in the proof of Theorem 3.2, by the curves $(\eta(s), \tau(s))$, with $s \geq 0$, where $\tau(s) := \bar{t} - \int_0^s (1 + l(r)) dr$.

A further remark on the modifications of the proof is the use of the following lemma in place of [22, Lemma 5.5].

Lemma 3.6. *Let $t > 0$, $x \in \bar{\Omega}$, $\psi \in C(\bar{\Omega} \times [0, t], \mathbb{R}^n)$ and $\varepsilon > 0$. Then there is a triple $(\eta, v, l) \in \text{SP}(x)$ such that for a.e. $s \in (0, t)$,*

$$H(\eta(s), \psi(\eta(s), \tau(s))) + L(\eta(s), -v(s)) \leq \varepsilon - v(s) \cdot \psi(\eta(s), \tau(s)),$$

where $\tau(s) := t - \int_0^s (1 + l(r)) dr$ and $\psi(x, s) := \psi(x, 0)$ for $s \leq 0$.

The above lemma can be proved in a parallel fashion as in the proof of [22, Lemma 5.5], and we leave it to the reader to prove the lemma.

3.2. Extremal curves or optimal controls. In this section we establish the existence of extremal curves (or optimal controls) $(\eta, v, l) \in \text{SP}$ for the variational formula (3.5). We set $Q = \bar{\Omega} \times (0, \infty)$.

Theorem 3.7. *Let $u_0 \in \text{Lip}(\bar{\Omega})$ and let $u \in \text{Lip}(Q)$ be the unique solution of (CN). Let $(x, t) \in Q$. Then there exists a triple $(\eta, v, l) \in \text{SP}_t(x)$ such that*

$$u(x, t) = \int_0^t (L(\eta(s), -v(s)) + f(s)) ds + u_0(\eta(t)),$$

where $f = F(\eta, v, l)$. Moreover, $\eta \in \text{Lip}([0, t], \mathbb{R}^n)$ and $(v, l, f) \in L^\infty([0, t], \mathbb{R}^{n+2})$.

Proof. Fix $(x, t) \in \bar{Q}$. In view of formula (3.5), we may choose a sequence $\{(\eta_k, v_k, l_k)\} \subset \text{SP}_t(x)$ such that for $k \in \mathbb{N}$,

$$u(x, t) + \frac{1}{k} > \int_0^t (L(\eta_k(s), -v_k(s)) + f_k(s)) ds + u_0(\eta_k(t)), \quad (3.16)$$

where $f_k := F(\eta_k, v_k, l_k)$.

We show that the sequence $\{v_k\}$ is uniformly integrable on $[0, t]$. Once this is done, due to Lemma 3.4, the sequences $\{\dot{\eta}_k\}$ and $\{l_k\}$ are also uniformly integrable on $[0, t]$. If we choose a constant $C_0 > 0$ so that $C_0 \geq \max_{\partial\Omega} B(x, 0)$, then $G(x, \xi) \geq -C_0$ for all $(x, \xi) \in \partial\Omega \times \mathbb{R}^n$ and hence, $f_k(s) \geq -C_0 l_k(s)$ for a.e. $s \in [0, t]$. Due to Lemma 3.4, there is a constant $C_1 > 0$, independent of k , such that $f_k(s) \geq -C_1 |v_k(s)|$ for a.e. $s \in [0, t]$, which implies that $|f_k(s)| \leq f_k(s) + 2C_1 |v_k(s)|$ for a.e. $s \in [0, t]$. Since H is coercive, for each $A \geq 0$ there exists a constant $C(A) > 0$ such that $L(x, \xi) \geq A|\xi| - C(A)$ for all $(x, \xi) \in \bar{\Omega} \times \mathbb{R}^n$. Combining these two estimates, we get for all $A \geq 0$,

$$A|v_k(s)| + |f_k(s)| \leq L(\eta_k(s), -v_k(s)) + f_k(s) + C(2C_1 + A). \quad (3.17)$$

We fix any $A > 0$ and measurable $E \subset [0, t]$, and, using the above estimate with $A = 0$ and $A = A$, observe that

$$\begin{aligned} & \int_E (L(\eta_k(s), -v_k(s)) + f_k(s) + C(C_1)) \, ds \\ & \leq \int_0^t (L(\eta_k(s), -v_k(s)) + f_k(s) + C(C_1)) \, ds \\ & \leq u(x, t) - u_0(\eta_k(t)) + \frac{1}{k} + C(C_1)t, \end{aligned}$$

and hence,

$$\begin{aligned} A \int_E |v_k(s)| \, ds & \leq \int_E (L(\eta_k(s), -v_k(s)) + f_k(s)) \, ds + C(2C_1 + A)|E| \\ & \leq 2 \max_{\bar{\Omega} \times [0, t]} |u| + 1 + C(C_1)t + C(2C_1 + A)|E|, \end{aligned}$$

where $|E|$ denotes the Lebesgue measure of E . From this, we easily deduce that $\{v_k\}$ is uniformly integrable on $[0, t]$. Thus, the sequences $\{\dot{\eta}_k\}$, $\{v_k\}$ and $\{l_k\}$ are uniformly integrable on $[0, t]$.

Next, we show that $\{f_k\}$ is uniformly integrable on $[0, t]$. To this end, we fix two finite sequences $\{\alpha_j\}$ and $\{\beta_j\}$ so that

$$0 \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_m < \beta_m \leq t.$$

Set $\beta_0 = 0$ and $\alpha_{m+1} = t$. In view of the dynamic programming principle, we have for $j = 0, 1, 2, \dots, m$,

$$u(\eta_k(\beta_j), t - \beta_j) \leq u(\eta_k(\alpha_{j+1}), t - \alpha_{j+1}) + \int_{\beta_j}^{\alpha_{j+1}} (L(\eta_k(s), -v_k(s)) + f_k(s)) \, ds.$$

Subtracting these from (3.16) yields

$$\begin{aligned} \frac{1}{k} - \sum_{j=1}^m u(\eta_k(\beta_j), t - \beta_j) & > - \sum_{j=1}^m u(\eta_k(\alpha_j), t - \alpha_j) \\ & + \sum_{j=1}^m \int_{\alpha_j}^{\beta_j} (L(\eta_k(s), -v_k(s)) + f_k(s)) \, ds. \end{aligned} \quad (3.18)$$

Hence, if $K > 0$ is a Lipschitz bound of u , then we get

$$\sum_{j=1}^m \int_{\alpha_j}^{\beta_j} (L(\eta_k(s), -v_k(s)) + f_k(s)) \, ds \leq \frac{1}{k} + K \sum_{j=1}^m (|\eta_k(\beta_j) - \eta_k(\alpha_j)| + |\beta_j - \alpha_j|).$$

Now, using (3.17) with $A = 0$, we find that

$$\sum_{j=1}^m \int_{\alpha_j}^{\beta_j} |f_k(s)| \, ds \leq \frac{1}{k} + \sum_{j=1}^m \int_{\alpha_j}^{\beta_j} (K|\dot{\eta}_k(s)| + K + C(2C_1)) \, ds,$$

from which we infer that $\{f_k\}$ is uniformly integrable on $[0, t]$.

We apply the Dunford-Pettis theorem to the sequence $\{(\dot{\eta}_k, v_k, l_k, f_k)\}$, to find an increasing sequence $\{k_j\} \subset \mathbb{N}$ and functions $h, v \in L^1([0, t], \mathbb{R}^n)$, $l, f \in L^1([0, t], \mathbb{R})$ such that, as $j \rightarrow \infty$, $(\dot{\eta}_{k_j}, v_{k_j}, l_{k_j}, f_{k_j}) \rightarrow (h, v, l, f)$ weakly in $L^1([0, t], \mathbb{R}^{2n+2})$. Setting $\eta(s) = x + \int_0^s h(r) \, dr$ for $s \in [0, t]$, we have $\eta_{k_j}(s) \rightarrow \eta(s)$ uniformly on $[0, t]$ as $j \rightarrow \infty$. Then, as in the last half of the proof of [22, Lemma 7.1], we infer that

$$\int_0^t L(\eta(s), -v(s)) \, ds \leq \liminf_{j \rightarrow \infty} \int_0^t L(\eta_{k_j}(s), -v_{k_j}(s)) \, ds.$$

It is now obvious that

$$u(x, t) \geq \int_0^t (L(\eta(s), -v(s)) + f(s)) \, ds + u_0(\eta(t)). \quad (3.19)$$

Now, we show that $(\eta, v, l) \in \text{SP}_t(x)$. It is clear that $\eta(s) \in \bar{\Omega}$ for all $s \in [0, t]$ and $l(s) \geq 0$ for a.e. $s \in [0, t]$. It is thus enough to show that $(v(s) - \dot{\eta}(s), f(s)) \in l(s)\mathcal{G}(\eta(s))$ for a.e. $s \in [0, t]$. Setting $\xi_k := v_k - \dot{\eta}_k$ and $\xi := v - \dot{\eta}$ on $[0, t]$, we have

$$l_k(s)B(\eta_k(s), p) \geq \xi_k(s) \cdot p - f_k(s) \quad \text{for all } p \in \mathbb{R}^n \text{ and a.e. } s \in [0, t],$$

Let $\phi \in C([0, t], \mathbb{R})$ satisfy $\phi(s) \geq 0$ for all $s \in [0, t]$. We have

$$\int_0^t \phi(s) (l_k(s)B(\eta_k(s), p) - \xi_k(s) \cdot p + f_k(s)) \, ds \geq 0 \quad \text{for all } p \in \mathbb{R}^n.$$

Sending $k \rightarrow \infty$ along the subsequence $k = k_j$, we find that

$$\int_0^t \phi(s) (l(s)B(\eta(s), p) - \xi(s) \cdot p + f(s)) \, ds \geq 0 \quad \text{for } p \in \mathbb{R}^n.$$

This implies that $(\xi(s), f(s)) \in l(s)\mathcal{G}(\eta(s))$ for a.e. $s \in [0, t]$, and conclude that $(\eta, v, l) \in \text{SP}_t(x)$.

Next, we set $\tilde{f}(s) = F(\eta, v, l)(s)$ for $s \in [0, t]$. Since $(v(s) - \dot{\eta}(s), f(s)) \in l(s)\mathcal{G}(\eta(s))$ for a.e. $s \in [0, t]$, we see that $\tilde{f}(s) \leq f(s)$ for a.e. $s \in [0, t]$ and $\tilde{f} \in L^1([0, t], \mathbb{R})$. Using (3.19) and (3.5), we get

$$\begin{aligned} u(x, t) &\geq \int_0^t (L(\eta(s), -v(s)) + f(s)) \, ds + u_0(\eta(t)) \\ &\geq \int_0^t (L(\eta(s), -v(s)) + \tilde{f}(s)) \, ds + u_0(\eta(t)) \geq u(x, t). \end{aligned}$$

Therefore, we have $f(s) = \tilde{f}(s)$ for a.e. $s \in [0, t]$ and

$$u(x, t) = \int_0^t (L(\eta(s), -v(s)) + f(s)) \, ds + u_0(\eta(t)).$$

Finally, we check the regularity of the triple $(\eta, v, l) \in \text{SP}_t(x)$ and the function f . Fix any interval $[\alpha, \beta] \subset [0, t]$, and observe as in (3.18) that

$$\begin{aligned} \int_\alpha^\beta (L(\eta(s), -v(s)) + f(s)) \, ds &\leq u(\eta(\beta), t - \beta) - u(\eta(\alpha), t - \alpha) \\ &\leq K \int_\alpha^\beta |\dot{\eta}(s)| \, ds + K(\beta - \alpha). \end{aligned}$$

Here we may choose a constant $C_3 > 0$ so that $|\dot{\eta}(s)| \leq C_3|v_k(s)|$ for a.e. $s \in [0, t]$. Combining the above and (3.16), with (η, v, f) in place of (η_k, v_k, f_k) and $A = KC_3 + 1$, and setting $C_4 = C(2C_1 + KC_3 + 1)$, we get

$$\int_\alpha^\beta (|v(s)| + |f(s)|) \, ds \leq (K + C_4)(\beta - \alpha),$$

from which we conclude that $(v, f) \in L^\infty([0, t], \mathbb{R}^{n+1})$ as well as $(\dot{\eta}, l) \in L^\infty([0, t], \mathbb{R}^{n+1})$. \square

An immediate consequence of the previous theorem is the following.

Theorem 3.8. *Let $\phi \in \text{Lip}(\bar{\Omega})$ be a solution of (E1), with $a = 0$. Let $x \in \bar{\Omega}$. Then there is a triple $(\eta, v, l) \in \text{SP}(x)$ such that for any $t > 0$,*

$$\phi(x) - \phi(\eta(t)) = \int_0^t (L(\eta(s), -v(s)) + f(s)) \, ds, \quad (3.20)$$

where $f := F(\eta, v, l)$. Moreover, $\eta \in \text{Lip}([0, \infty), \mathbb{R}^n)$ and $(v, l, f) \in L^\infty([0, \infty), \mathbb{R}^{n+2})$.

Proof. Note that the function $u(x, t) := \phi(x)$ is a solution of (CN). Using Theorem 3.7, we define inductively the sequence $\{(\eta_k, v_k, l_k)\}_{k \geq 0} \subset \text{SP}$ as follows. We first choose a $(\eta_0, v_0, l_0) \in \text{SP}(x)$ so that

$$\phi(\eta_0(0)) - \phi(\eta_0(1)) = \int_0^1 (L(\eta_0(s), -v_0(s)) + F(\eta_0, v_0, l_0)(s)) \, ds.$$

Next, we assume that $\{(\eta_k, v_k, l_k)\}_{k \leq j-1}$, with $j \geq 1$, is given, and choose a $(\eta_j, v_j, l_j) \in \text{SP}(\eta_{j-1}(1))$ so that

$$\phi(\eta_j(1)) - \phi(\eta_j(0)) = \int_0^1 (L(\eta_j(s), -v_j(s)) + F(\eta_j, v_j, l_j)(s)) \, ds.$$

Once the sequence $\{(\eta_k, v_k, l_k)\}_{k \geq 0} \subset \text{SP}$ is given, we define the $(\eta, v, l) \in \text{SP}(x)$ and the function f on $[0, \infty)$ by setting for $k \in \mathbb{N} \cup \{0\}$ and $s \in [0, 1]$, $(\eta(s + k), v_k(s + k), l(s + k), f(s + k)) = (\eta_k(s), v_k(s), l_k(s), F(\eta_k, v_k, l_k)(s))$.

It is clear that $(\eta, v, l) \in \text{SP}(x)$, $f = F(\eta, v, l)$ and (3.20) is satisfied. Thanks to Theorem 3.7, we have $\eta_k \in \text{Lip}([0, 1], \mathbb{R}^n)$ and $(v_k, l_k, f_k) \in L^\infty([0, 1], \mathbb{R}^{n+2})$ for $k \geq 0$. Moreover, in view of the proof of Theorem 3.7, we see easily

that $\sup_{k \geq 0} \|(v_k, f_k)\|_{L^\infty([0,1])} < \infty$, from which we conclude that $(\eta, v, l, f) \in \text{Lip}([0, \infty), \mathbb{R}^n) \times L^\infty([0, \infty), \mathbb{R}^{n+2})$. \square

3.3. Derivatives of subsolutions along curves. Throughout this section we fix a subsolution $u \in \text{USC}(\bar{\Omega})$ of (E1) with $a = 0$, $0 < T < \infty$ and a Lipschitz curve η in $\bar{\Omega}$, i.e., $\eta \in \text{Lip}([0, T], \mathbb{R}^n)$ and $\eta([0, T]) \subset \bar{\Omega}$.

Henceforth in this section we assume that there is a bounded, open neighborhood V of $\partial\Omega$ for which H, B and n are defined and continuous on $(\Omega \cup \bar{V}) \times \mathbb{R}^n$, $\bar{V} \times \mathbb{R}^n$ and \bar{V} , respectively. Moreover, we assume by replacing θ and M_B in (A3), (A4) respectively by other positive numbers if needed that (A1), with V in place of Ω , and (A3)–(A5), with V in place of $\partial\Omega$, are satisfied. (Of course, these are not real additional assumptions.)

Theorem 3.9. *There exists a function $p \in L^\infty([0, T], \mathbb{R}^n)$ such that for a.e. $t \in [0, T]$, $\frac{d}{dt}u \circ \eta(t) = p(t) \cdot \dot{\eta}(t)$, $H(\eta(t), p(t)) \leq 0$, and $B(\eta(t), p(t)) \leq 0$ if $\eta(t) \in \partial\Omega$.*

To prove the above theorem, we use the following lemmas.

Lemma 3.10. *Let $w \in \text{Lip}(\bar{\Omega})$, $\{w_\varepsilon\}_{\varepsilon > 0} \subset \text{Lip}(\bar{\Omega})$ and $\{p_\varepsilon\}_{\varepsilon > 0} \subset L^\infty([0, T], \mathbb{R}^n)$. Assume that $w_\varepsilon(x) \rightarrow w(x)$ uniformly on $\bar{\Omega}$ as $\varepsilon \rightarrow 0$ and, for a.e. $t \in [0, T]$,*

$$\begin{cases} \frac{d}{dt}w_\varepsilon \circ \eta(t) = p_\varepsilon(t) \cdot \dot{\eta}(t), & H(\eta(t), p_\varepsilon(t)) \leq \varepsilon, \\ B(\eta(t), p_\varepsilon(t)) \leq \varepsilon & \text{if } \eta(t) \in \partial\Omega. \end{cases} \quad (3.21)$$

then there exists a function $p \in L^\infty([0, T], \mathbb{R}^n)$ such that for a.e. $t \in [0, T]$, $\frac{d}{dt}w \circ \eta(t) = p(t) \cdot \dot{\eta}(t)$, $H(\eta(t), p(t)) \leq 0$, and $B(\eta(t), p(t)) \leq 0$ if $\eta(t) \in \partial\Omega$.

Proof. Observe first that for all $t \in [0, T]$,

$$\left| w(\eta(t)) - w(\eta(0)) - \int_0^t p_\varepsilon(s) \cdot \dot{\eta}(s) ds \right| \leq 2\|w_\varepsilon - w\|_{L^\infty(\Omega)}.$$

Next, we observe by the coercivity of H that $\{p_\varepsilon\}$ is bounded in $L^\infty([0, T], \mathbb{R}^n)$, and then, in view of the Banach-Sack theorem, we may choose a sequence $\{p_j\}_{j \in \mathbb{N}}$ and a function $p \in L^\infty([0, T], \mathbb{R}^n)$ so that p_j is in the closed convex hull of $\{p_\varepsilon : 0 < \varepsilon < 1/j\}$, $p_j \rightarrow p$ strongly in $L^2([0, T], \mathbb{R}^n)$ as $j \rightarrow \infty$ and $p_j(t) \rightarrow p(t)$ for a.e. $t \in [0, T]$ as $j \rightarrow \infty$. By (3.21) and the convexity of H and B , we see that, for a.e. $t \in [0, T]$, $H(\eta(t), p_j(t)) \leq j^{-1}$, and $B(\eta(t), p_j(t)) \leq j^{-1}$ if $\eta(t) \in \partial\Omega$. Moreover, we have, for all $t \in [0, T]$,

$$\left| w(\eta(t)) - w(\eta(0)) - \int_0^t p_j(s) \cdot \dot{\eta}(s) ds \right| \leq 2 \sup_{0 < \varepsilon < j^{-1}} \|w_\varepsilon - w\|_{L^\infty(\Omega)}.$$

Now, by sending $j \rightarrow \infty$, we get for a.e. $t \in [0, T]$, $H(\eta(t), p(t)) \leq 0$, and $B(\eta(t), p(t)) \leq 0$ if $\eta(t) \in \partial\Omega$, and, for all $t \in [0, T]$, $w(\eta(t)) - w(\eta(0)) = \int_0^t p(s) \cdot \dot{\eta}(s) ds$. The proof is complete. \square

Lemma 3.11. *Let $z \in \partial\Omega$ and $\varepsilon > 0$. Then there are an open neighborhood U of z in $\bar{\Omega}$, a sequence $\{V_j\}_{j \in \mathbb{N}}$ of open neighborhoods of $U \cap \partial\Omega$ in V and a*

sequence $\{u_j\}_{j \in \mathbb{N}}$ of C^1 functions on $W_j := U \cup V_j$, such that for each j the function u_j satisfies

$$\begin{cases} H(x, Du_j(x)) \leq \varepsilon & \text{in } W_j, \\ B(x, Du_j(x)) \leq \varepsilon & \text{in } V_j, \end{cases}$$

and, as $j \rightarrow \infty$, $u_j(x) \rightarrow u(x)$ uniformly on U .

We now prove Theorem 3.9 by assuming Lemma 3.11, the proof of which will be given after the proof of Theorem 3.9.

Proof of Theorem 3.9. In view of Lemma 3.10, it is enough to show that for each $\varepsilon > 0$ there exists a function $p_\varepsilon \in L^\infty([0, T], \mathbb{R}^n)$ such that for a.e. $t \in [0, T]$, $\frac{d}{dt}u \circ \eta(t) = p_\varepsilon(t) \cdot \dot{\eta}(t)$, $H(\eta(t), p_\varepsilon(t)) \leq \varepsilon$, and $B(\eta(t), p_\varepsilon(t)) \leq \varepsilon$ if $\eta(t) \in \partial\Omega$. To show this, we fix any $\varepsilon > 0$. It is sufficient to prove that for each $\tau \in [0, T]$, there exist a neighborhood I_τ of τ , relative to $[0, T]$, and a function $p_\tau \in L^\infty(I_\tau, \mathbb{R}^n)$ such that for a.e. $t \in I_\tau$, $\frac{d}{dt}u \circ \eta(t) = p_\tau(t) \cdot \dot{\eta}(t)$, $H(\eta(t), p_\tau(t)) \leq \varepsilon$, and $B(\eta(t), p_\tau(t)) \leq \varepsilon$ if $\eta(t) \in \partial\Omega$.

Fix any $\tau \in [0, T]$. Consider first the case where $\eta(\tau) \in \Omega$. There is a $\delta > 0$ such that $\eta(I_\tau) \subset \Omega$, where $I_\tau := [\tau - \delta, \tau + \delta] \cap [0, T]$. We may choose an open neighborhood U of z such that $\eta(I_\tau) \subset U \Subset \Omega$. By the mollification technique, for any $\alpha > 0$, we may choose a function $u_\alpha \in C^1(U)$ such that $H(x, Du_\alpha(x)) \leq \varepsilon$ and $|u_\alpha(x) - u(x)| < \alpha$ for all $x \in U$. Then, setting $p_{\tau, \alpha}(t) = Du_\alpha(\eta(t))$ for $t \in I_\tau$ and $\alpha > 0$, we have $\frac{d}{dt}u_\alpha \circ \eta(t) = p_{\tau, \alpha}(t) \cdot \dot{\eta}(t)$ and $H(\eta(t), p_{\tau, \alpha}(t)) \leq \varepsilon$ for a.e. $t \in I_\tau$ and all $\alpha > 0$. Hence, by Lemma 3.10, we find that there is a function $p_\tau \in L^\infty(I_\tau, \mathbb{R}^n)$ such that for a.e. $t \in I_\tau$, $\frac{d}{dt}u \circ \eta(t) = p_\tau(t) \cdot \dot{\eta}(t)$ and $H(\eta(t), p_\tau(t)) \leq \varepsilon$.

Next consider the case where $\eta(\tau) \in \partial\Omega$. Thanks to Lemma 3.11, there are an open neighborhood U of $\eta(\tau)$ in $\bar{\Omega}$, a sequence $\{V_j\}_{j \in \mathbb{N}}$ of open neighborhoods of $U \cap \partial\Omega$ in V and a sequence $\{u_j\}_{j \in \mathbb{N}}$ of C^1 functions on $W_j := U \cup V_j$ such that for any $j \in \mathbb{N}$, $H(x, Du_j(x)) \leq \varepsilon$ for all $x \in W_j$, $B(x, Du_j(x)) \leq \varepsilon$ for all $x \in V_j$ and $|u_j(x) - u(x)| < 1/j$ for all $x \in U$. We now choose a constant $\delta > 0$ so that if $I_\tau := [\tau - \delta, \tau + \delta] \cap [0, T]$, then $\eta(I_\tau) \subset U$. Set $p_j(t) = Du_j(\eta(t))$ for $t \in I_\tau$. Then, for a.e. $t \in [0, T]$, we have $\frac{d}{dt}u_j \circ \eta(t) = p_j(t) \cdot \dot{\eta}(t)$, $H(\eta(t), p_j(t)) \leq \varepsilon$, and $B(\eta(t), p_j(t)) \leq \varepsilon$ if $\eta(t) \in \partial\Omega$. Lemma 3.10 now ensures that there exists a function $p_\tau \in L^\infty(I_\tau, \mathbb{R}^n)$ such that for a.e. $t \in I_\tau$, $\frac{d}{dt}u \circ \eta(t) = p_\tau(t) \cdot \dot{\eta}(t)$, $H(\eta(t), p_\tau(t)) \leq \varepsilon$, and $B(\eta(t), p_\tau(t)) \leq \varepsilon$ if $\eta(t) \in \partial\Omega$. The proof is now complete. \square

For the proof of Lemma 3.11, we need the following lemma.

Lemma 3.12. *Let $w \in \text{Lip}(\bar{\Omega})$ be a subsolution of (E1), with $a = 0$. Let $z \in \partial\Omega$ and $p \in D^+w(z)$. Assume that $p + tn(z) \notin D^+w(z)$ for all $t > 0$. Then*

$$p \in \bigcap_{r>0} \overline{\bigcup_{x \in B_r(z) \cap \Omega} D^+w(x)}.$$

In particular, we have $H(z, p) \leq 0$.

Proof. We choose a function $\phi \in C^1(\bar{\Omega})$ so that $D\phi(z) = p$ and the function $w - \phi$ attains a strict maximum at z . Let $\psi \in C^1(\mathbb{R}^n)$ be a function such that $\Omega = \{x \in \mathbb{R}^n : \psi(x) < 0\}$ and $D\psi(x) \neq 0$ for all $x \in \partial\Omega$. For $\varepsilon > 0$, let $x_\varepsilon \in \bar{\Omega}$ be a maximum point of the function $\Phi := w - \phi - \varepsilon\psi$ on $\bar{\Omega}$. It is obvious that $x_\varepsilon \rightarrow z$ as $\varepsilon \rightarrow 0+$ and $D(\phi + \varepsilon\psi)(x_\varepsilon) \in D^+w(x_\varepsilon)$. Suppose that $x_\varepsilon = z$. Then we have $D\phi(z) + \varepsilon|D\psi(z)|n(z) \in D^+w(z)$, which is impossible by the choice of p . That is, we have $x_\varepsilon \neq z$. Observe that for any $x \in \partial\Omega$, $\Phi(x) = (w - \phi)(x) \leq (w - \phi)(z) = \Phi(z) < \Phi(x_\varepsilon)$, which guarantees that $x_\varepsilon \in \Omega$. Thus we have $p = \lim_{\varepsilon \rightarrow 0+} D(\phi + \varepsilon\psi)(x_\varepsilon) \in \bigcup_{r>0} \overline{\bigcap_{x \in \Omega \cap B_r(z)} D^+w(x)}$, which implies that $H(z, p) \leq 0$. \square

Proof of Lemma 3.11. We fix any $0 < \varepsilon < 1$ and $z \in \partial\Omega$. Since Ω is a C^1 domain, we may assume after a change of variables if necessary that $z = 0$ and for some constant $r > 0$,

$$B_r \cap \bar{\Omega} = \{x = (x_1, \dots, x_n) \in B_r : x_n \leq 0\}.$$

Of course, we have $\tilde{n}(x) = n(z) = e_n$ for all $x \in B_r \cap \partial\Omega$.

Now, we choose a constant $K > 0$ so that for $(x, p) \in \bar{\Omega} \times \mathbb{R}^n$, if $H(x, p) \leq 0$, then $|p| \leq K$. We next choose a constant $R > 0$ so that $|B(x, p) - B(z, p)| \leq \varepsilon$ for all $(x, p) \in (\partial\Omega \cap B_R) \times B_K$. Replacing r by R if $r > R$, we may assume that $r \leq R$.

We now show that u is a subsolution of

$$\begin{cases} H(x, Du(x)) \leq 0 & \text{in } B_r \cap \Omega, \\ B(z, Du(x)) \leq \varepsilon & \text{on } B_r \cap \partial\Omega. \end{cases} \quad (3.22)$$

To do this, we fix any $x \in B_r \cap \bar{\Omega}$ and $p \in D^+u(x)$. We need to consider only the case when $x \in \partial\Omega$. We may assume that $H(x, p) > 0$. Otherwise, we have nothing to prove. We set $\tau := \sup\{t \geq 0 : p + t\tilde{n}(x) \in D^+u(x)\}$. Note that $B(x, p) \leq 0$ and, therefore, $\tau \geq 0$. Since the function $t \mapsto B(x, p + t\tilde{n}(x)) - \theta t$ is non-decreasing on \mathbb{R} , we see that $B(x, p + t\tilde{n}(x)) > 0$ for all t large enough. Also, it is obvious that $H(x, p + t\tilde{n}(x)) > 0$ for all t large enough. Therefore, we see that $p + t\tilde{n}(x) \notin D^+u(x)$ if t is large enough and conclude that $0 \leq \tau < \infty$.

Since $D^+u(x)$ is a closed subset of \mathbb{R}^n , we see that $p + \tau\tilde{n}(x) \in D^+u(x)$. From the definition of τ , we observe that $p + t\tilde{n}(x) \notin D^+u(x)$ for $t > \tau$. We now invoke Lemma 3.12, to find that $H(x, p + \tau\tilde{n}(x)) \leq 0$.

We recall the standard observation that if $q \in D^+u(x)$, then $q - t\tilde{n}(x) \in D^+u(x)$ for all $t \geq 0$. Hence, we must have either $H(x, p + t\tilde{n}(x)) \leq 0$ or $B(x, p + t\tilde{n}(x)) \leq 0$ for any $t \leq \tau$. Set $\sigma := \sup\{t \in [0, \tau] : H(x, p + t\tilde{n}(x)) > 0\}$, and observe that $0 < \sigma \leq \tau$ and $H(x, p + \sigma\tilde{n}(x)) \leq 0$. There is a sequence $\{t_j\} \subset [0, \tau]$ converging to σ such that $H(x, p + t_j\tilde{n}(x)) > 0$, which implies that $B(x, p + t_j\tilde{n}(x)) \leq 0$. Hence we have $B(x, p + \sigma\tilde{n}(x)) \leq 0$. Thus we have $\sigma > 0$, $H(x, p + \sigma\tilde{n}(x)) \leq 0$ and $B(x, p + \sigma\tilde{n}(x)) \leq 0$. By the choice of K , we have $p + \sigma n(z) \in B_K$, and hence $B(z, p + \sigma\tilde{n}(x)) \leq \varepsilon$. Noting that $\tilde{n}(x) = \tilde{n}(z)$ and $\sigma > 0$, we see by the monotonicity of $t \mapsto B(z, p + tn(z))$ that $B(z, p) \leq \varepsilon$. Thus we find that u is a subsolution of (3.22).

Following the arguments of Lemma 4.3, we can show that there exist a function $\zeta \in C^1(\mathbb{R}^n) \cap \text{Lip}(\mathbb{R}^n)$ and a constant $\delta > 0$ such that for all $\xi \in \mathbb{R}^n$, $\zeta(\xi) \geq (K+1)|\xi|$ and

$$B(z, D\zeta(\xi)) - 3\varepsilon \begin{cases} > -\varepsilon & \text{if } \xi_n \geq -\delta, \\ < \varepsilon & \text{if } \xi_n \leq \delta. \end{cases}$$

We may also assume that $\zeta \in C^\infty(\mathbb{R}^n)$ and all the derivatives of ζ are bounded on \mathbb{R}^n .

We introduce the sup-convolution of u as follows:

$$u_\alpha(x) := \max_{y \in \bar{B}_r \cap \bar{\Omega}} \left(u(y) - \alpha \zeta\left(\frac{y-x}{\alpha}\right) \right) \quad \text{for } x \in \mathbb{R}^n,$$

where $\alpha > 0$. We write $\zeta_\alpha(\xi)$ for $\alpha\zeta(\xi/\alpha)$ for convenience, and note that $B(z, D\zeta_\alpha(\xi)) > 2\varepsilon$ if $\xi_n \geq -\alpha\delta$ and $B(z, D\zeta_\alpha(\xi)) < 4\varepsilon$ if $\xi_n \leq \alpha\delta$. Set $U = B_{r/2} \cap \{x \in \mathbb{R}^n : x_n \leq 0\}$, $V_\alpha = \{x \in B_{r/2} : |x_n| < \delta\alpha^2\}$ and $W_\alpha := \{x \in B_{r/2} : x_n < \delta\alpha^2\}$. Note that U is an open neighborhood of $z = 0$ relative to $\bar{\Omega}$, V_α is a open neighborhood of $U \cap \partial\Omega$ and $W_\alpha = U \cup V_\alpha$. We choose an $0 < \alpha_0 < 1$ so that $V_\alpha \subset V$ for all $0 < \alpha < \alpha_0$, and assume henceforth that $0 < \alpha < \alpha_0$.

We now prove that if α is small enough, then u_α satisfies in the viscosity sense

$$\begin{cases} H(x, Du_\alpha(x)) \leq \varepsilon & \text{in } W_\alpha, \\ B(z, Du_\alpha(x)) \leq 4\varepsilon & \text{in } V_\alpha. \end{cases}$$

To this end, we fix any $\hat{x} \in W_\alpha$ and $\hat{p} \in D^+u_\alpha(\hat{x})$. Choose $\hat{y} \in \bar{B}_r \cap \bar{\Omega}$ so that $u_\alpha(\hat{x}) = u(\hat{y}) - \zeta_\alpha(\hat{y} - \hat{x})$. It is a standard observation that if $\hat{y} \in B_r$, then $D\zeta_\alpha(\hat{y} - \hat{x}) = \hat{p} \in D^+u(\hat{y})$.

Next, let \bar{x} denote the projection of \hat{x} on the half space $\{x \in \mathbb{R}^n : x_n \leq 0\}$. That is, $\bar{x} = \hat{x}$ if $\hat{x}_n \leq 0$ and $\bar{x} = (\hat{x}_1, \dots, \hat{x}_{n-1}, 0)$ otherwise. We note that $|\bar{x} - \hat{x}| < \delta\alpha^2 < \delta\alpha < \delta$ and $u(\bar{x}) - \zeta_\alpha(\bar{x} - \hat{x}) \leq u_\alpha(\hat{x}) = u(\hat{y}) - \zeta_\alpha(\hat{y} - \hat{x})$. Hence,

$$u(\hat{y}) - u(\bar{x}) \geq \zeta_\alpha(\hat{y} - \hat{x}) - \zeta_\alpha(\bar{x} - \hat{x}) \geq (K+1)|\hat{y} - \hat{x}| - \alpha \sup_{\xi \in B_{\delta\alpha^2}} \zeta(\xi/\alpha),$$

and furthermore, $(K+1)|\hat{y} - \hat{x}| \leq K|\hat{x} - \hat{y}| + R\alpha$, where $R := K\delta + \sup_{\xi \in B_\delta} \zeta(\xi)$. Accordingly, we get $|\hat{y} - \hat{x}| \leq R\alpha$. We may assume that $R\alpha_0 < r/2$, so that $\hat{y} \in B_r$.

If $\hat{y}_n < 0$, then $\hat{y} \in \Omega$ and we have $H(\hat{y}, \hat{p}) \leq 0$. Moreover, writing ω_H for the modulus of H on $(B_r \cap \bar{\Omega}) \times B_K$, we get $H(\hat{x}, \hat{p}) \leq H(\hat{y}, \hat{p}) + \omega_H(R\alpha) \leq \omega_H(R\alpha)$. We may assume by reselecting α_0 by a smaller positive number that $\omega_H(R\alpha) < \varepsilon$. Thus we have $H(\hat{x}, \hat{p}) \leq \varepsilon$.

Next, assume that $\hat{y}_n = 0$. We have $\hat{y}_n - \bar{x}_n \geq 0$, and hence, $B(z, D\zeta_\alpha(\hat{y} - \bar{x})) > 2\varepsilon$. Since $|\bar{x} - \hat{x}| < \delta\alpha^2$, we find that $|D\zeta_\alpha(\hat{y} - \hat{x}) - D\zeta_\alpha(\hat{y} - \bar{x})| \leq C \frac{|\bar{x} - \hat{x}|}{\alpha} \leq C\delta\alpha$, and

$$B(z, D\zeta_\alpha(\hat{y} - \hat{x})) \geq B(z, D\zeta_\alpha(\hat{y} - \bar{x})) - M_B C \delta \alpha > 2\varepsilon - M_B C \delta \alpha,$$

where $C > 0$ is a Lipschitz bound of $D\zeta$. We may assume by replacing α_0 by a smaller positive number if needed that $M_B C \delta \alpha < \varepsilon$. Then we have $B(z, \hat{p}) = B(z, D\zeta_\alpha(\hat{y} - \hat{x})) > \varepsilon$, and therefore, $H(\hat{y}, \hat{p}) \leq 0$. As before, we get $H(\hat{x}, \hat{p}) \leq \omega_H(R\alpha) \leq \varepsilon$. Thus we conclude that if $0 < \alpha < \alpha_0$, then $H(x, Du_\alpha(x)) \leq \varepsilon$ is satisfied in W_α in the viscosity sense.

Next, we assume that $\hat{x} \in V_\alpha$. Since $\hat{y}_n \leq 0$, we have $\hat{y}_n - \hat{x}_n < \delta \alpha^2 < \delta \alpha$ and $B(z, \hat{p}) = B(z, D\zeta_\alpha(\hat{y} - \hat{x})) < 4\varepsilon$. Thus, u_α satisfies $B(z, Du_\alpha(x)) \leq 4\varepsilon$ in V_α in the viscosity sense.

Since $H(x, Du_\alpha(x)) \leq \varepsilon$ in W_α in the viscosity sense, the functions u_α on W_α , with $0 < \alpha < \alpha_0$, have a common Lipschitz bound. Therefore, by replacing r a smaller positive number if necessary, we may assume that for any $0 < \alpha < \alpha_0$, $B(x, Du_\alpha(x)) \leq 5\varepsilon$ in V_α in the viscosity sense.

Finally, we fix $j \in \mathbb{N}$ and choose an $\alpha_j \in (0, \alpha_0)$ so that $|u_{\alpha_j}(x) - u(x)| < 1/j$ for all $x \in U$. By mollifying u_{α_j} , we may find a function $u_j \in C^1(\frac{1}{2}W_{\alpha_j})$ such that $|u_j(x) - u(x)| < 2/j$ for all $x \in \frac{1}{2}U$, $H(x, Du_j(x)) \leq 2\varepsilon$ for all $x \in \frac{1}{2}W_{\alpha_j}$ and $B(x, Du_j(x)) \leq 6\varepsilon$ for all $x \in \frac{1}{2}V_{\alpha_j}$. The collection of the open subset $\frac{1}{2}U$ of $\bar{\Omega}$, the sequence $\{\frac{1}{2}V_{\alpha_j}\}_{j \in \mathbb{N}}$ of neighborhoods of $\partial\Omega \cap \frac{1}{2}U$ and the sequence $\{u_j\}_{j \in \mathbb{N}}$ of functions gives us what we needed. \square

Lemma 3.13 (A convexity lemma). *Let $\{u_\lambda\}_{\lambda \in \Lambda} \subset C(\bar{\Omega} \times (0, \infty))$ be a nonempty collection of subsolutions of (3.6), (3.7). Set $u(x, t) = \inf_{\lambda \in \Lambda} u_\lambda(x, t)$ for $(x, t) \in \bar{\Omega} \times (0, \infty)$. Assume that u is a real-valued function on $\bar{\Omega} \times (0, \infty)$. Then u is a subsolution of (3.6), (3.7).*

Proof. Set $Q = \bar{\Omega} \times (0, \infty)$. Fix $(\hat{x}, \hat{t}) \in Q$ and $\phi \in C^1(Q)$, and assume that $u - \phi$ attains a strict maximum at (\hat{x}, \hat{t}) . We may assume that ϕ has the form: $\phi(x, t) = \psi(x) + \chi(t)$ for some functions ψ and χ . Fix any $\varepsilon > 0$. By Lemma 3.3, there exists $(\gamma, g) \in C(\partial\Omega, \mathbb{R}^{n+1})$ such that $(\gamma(x), g(x)) \in \mathcal{G}(x)$ and $B(x, D\psi(x)) < \varepsilon + \gamma(x) \cdot D\psi(x) - g(x)$ for all $x \in \partial\Omega$. By this first condition, we see that the functions u_λ , with $\lambda \in \Lambda$, are subsolutions of

$$\begin{cases} u_t(x, t) + H(x, Du(x, t)) = 0 & \text{in } Q, \\ \gamma(x) \cdot Du(x, t) = g(x) & \text{on } \partial\Omega \times (0, \infty). \end{cases} \quad (3.23)$$

By [22, Theorem 2.8], we find that u is a subsolution of (3.23), which implies that either $\chi_t(\hat{t}) + H(\hat{x}, D\psi(\hat{x})) \leq 0$, or $\hat{x} \in \partial\Omega$ and $\gamma(\hat{x}) \cdot D\psi(\hat{x}) - g(\hat{x}) \leq 0$. But, this last inequality guarantees that $B(\hat{x}, D\psi(\hat{x})) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we see that either $\chi_t(\hat{t}) + H(\hat{x}, D\psi(\hat{x})) \leq 0$, or $\hat{x} \in \partial\Omega$ and $B(\hat{x}, D\psi(\hat{x})) \leq 0$. Hence, u is a subsolution of (3.6), (3.7). \square

The above proof reduces the nonlinear boundary condition to the case of a family of linear Neumann conditions. One can prove the above lemma without such a linearization procedure by treating directly the nonlinear condition and using Lemma 3.11 as in the proof of [22, Theorem 2.8].

It is worthwhile to noticing that another convexity lemma is valid. That is, if u and v are subsolutions of (3.6), (3.7), then so is the function $\lambda u + (1 - \lambda)v$, with $0 < \lambda < 1$.

Note that the propositions, corresponding to this convexity lemma and Lemma 3.13, are valid for (DBC).

3.4. Convergence to asymptotic solutions. In this section, we give the second proof of Theorem 1.5 for (CN). We write $Q = \bar{\Omega} \times (0, \infty)$ throughout this section.

We define the function u_∞ on $\bar{\Omega}$ by

$$u_\infty(x) = \inf\{\phi(x) : \phi \in \text{Lip}(\bar{\Omega}), \phi \text{ is a solution of (E1) with } a = 0, \phi \geq u_0^- \text{ on } \bar{\Omega}\}, \quad (3.24)$$

where u_0^- is the function on $\bar{\Omega}$ given by

$$u_0^-(x) = \sup\{\psi(x) : \psi \in \text{Lip}(\bar{\Omega}), \psi \text{ is a subsolution of (E1) with } a = 0, \psi \leq u_0 \text{ on } \bar{\Omega}\}.$$

It is a standard observation that u_0 and u_∞ are Lipschitz continuous functions on $\bar{\Omega}$ and are, respectively, a subsolution and a supersolution of (E1), with $a = 0$. Moreover, using Lemma 3.13, we see that u_∞ is a solution of (E1), with $a = 0$.

Lemma 3.14. *We have $u_\infty(x) = \liminf_{t \rightarrow \infty} u(x, t)$ for all $x \in \bar{\Omega}$.*

For the proof of this lemma, we refer to the proof of [23, Proposition 4.4], which can be easily adapted to the present situation, and we skip it here.

Proof of Theorem 1.5. We show that

$$\lim_{t \rightarrow \infty} u(x, t) = u_\infty(x) \quad \text{uniformly for } x \in \bar{\Omega}. \quad (3.25)$$

Therefore, in order to prove (3.25), it is enough to show the pointwise convergence in (3.25). Moreover, by Lemma 3.14, we need only to show that

$$\limsup_{t \rightarrow \infty} u(x, t) \leq u_\infty(x) \quad \text{for all } x \in \bar{\Omega}.$$

Now, we fix any $x \in \bar{\Omega}$. Since u_∞ is a solution of (E1), with $a = 0$, by Theorem 3.8, there is an extremal triple $(\eta, v, l) \in \text{SP}(x)$ such that, if we set $f = F(\eta, v, l)$, then we have

$$u_\infty(\eta(0)) - u_\infty(\eta(t)) = \int_0^t (L(\eta(s), -v(s)) + f(s)) \, ds \quad \text{for all } t \geq 0,$$

which is equivalent to the condition that $-\frac{d}{dt}u_\infty \circ \eta(t) = L(\eta(t), -v(t)) + f(t)$ for a.e. $t \geq 0$. Moreover, we have $\dot{\eta}, v \in L^\infty([0, \infty), \mathbb{R}^n)$ and $l, f \in L^\infty([0, \infty), \mathbb{R})$. By Theorem 3.9, there exists a function $p \in L^\infty([0, \infty), \mathbb{R}^n)$ such that for a.e. $t \in [0, \infty)$, $\frac{d}{dt}u_\infty \circ \eta(t) = p(t) \cdot \dot{\eta}(t)$, $H(\eta(t), p(t)) \leq 0$ and $B(\eta(t), p(t)) \leq 0$ if $\eta(t) \in \partial\Omega$. Observe that $l(t)B(\eta(t), p(t)) \geq (v - \dot{\eta})(t) \cdot p(t) - f(t)$ for a.e. $t \in [0, \infty)$.

Next, combining the above relations together with the Fenchel-Young inequality, $-w \cdot q \leq H(y, q) + L(y, -w)$, we observe that if we set $\xi = v - \dot{\eta}$, then for a.e. $t \geq 0$,

$$\begin{aligned} -\frac{d}{dt}u_\infty(\eta(t)) &= -p(t) \cdot \dot{\eta}(t) = -p(t) \cdot (v(t) - \xi(t)) \\ &\leq H(\eta(t), p(t)) + L(\eta(t), -v(t)) + p(t) \cdot \xi(t) \\ &\leq L(\eta(t), -v(t)) + p(t) \cdot \xi(t) \\ &\leq L(\eta(t), -v(t)) + l(t)B(\eta(t), p(t)) + f(t) \\ &\leq L(\eta(t), -v(t)) + f(t) = -\frac{d}{dt}u_\infty(\eta(t)). \end{aligned}$$

Thus, all the inequalities above are indeed equalities. In particular, we find that $-v(t) \cdot p(t) = H(\eta(t), p(t)) + L(\eta(t), -v(t)) = L(\eta(t), -v(t))$ for a.e. $t \geq 0$, which shows that $H(\eta(t), p(t)) = 0$ and $-v(t) \in \partial_p H(\eta(t), p(t))$ for a.e. $t \geq 0$.

We here consider only the case when (A7)₊ is valid. It is left to the reader to check the other case when (A7)₋ holds.

The argument outlined below is parallel to the last half of the proof of [23, Theorem 1.3]. Since (A7)₊ is assumed, there exist a constant $\delta_0 > 0$ and a function $\omega_0 \in C([0, \infty))$ satisfying $\omega_0(0) = 0$ such that for any $0 < \delta < \delta_0$ and $(y, z) \in \bar{\Omega} \times \mathbb{R}^n$, if $H(y, q) = 0$ and $z \in \partial_p H(y, q)$ for some $q \in \mathbb{R}^n$, then

$$L(y, (1 + \delta)z) \leq (1 + \delta)L(y, z) + \delta\omega_0(\delta).$$

This ensures that for a.e. $t \geq 0$ and all $0 < \delta < \delta_0$,

$$L(\eta(t), -(1 + \delta)v(t)) \leq (1 + \delta)L(\eta(t), -v(t)) + \delta\omega_0(\delta). \quad (3.26)$$

We fix $\varepsilon > 0$, and note (see for instance the proof of [23, Theorem 1.3]) that there is a positive constant T_0 and, for each $y \in \bar{\Omega}$, a constant $0 < T(y) \leq T_0$ such that $u(y, T(y)) < u_\infty(y) + \varepsilon$.

We choose $t_0 > T_0$ so that $T_0/(t_0 - T_0) < \delta_0$. Fix any $t \geq t_0$, and set $y = \eta(t)$, $T = T(y)$, $S = t - T$ and $\delta = (t - S)/S$. Note that $\delta = T/(t - T) < \delta_0$ and $\delta \rightarrow 0$ as $t \rightarrow \infty$. We set $\eta_\delta(t) = \eta((1 + \delta)t)$, $v_\delta(t) = (1 + \delta)v((1 + \delta)t)$, $l_\delta(t) = (1 + \delta)l((1 + \delta)t)$ and $f_\delta(t) = (1 + \delta)f((1 + \delta)t)$ for $t \geq 0$. Using (3.26) and noting that $(1 + \delta)S = t$ and $\delta S = T \leq T_0$, we get

$$\int_0^S L(\eta_\delta(s), -v_\delta(s)) ds \leq \int_0^t L(\eta(s), -v(s)) ds + T_0\omega_0(\delta).$$

Hence, noting that $(\eta_\delta, v_\delta, l_\delta) \in \text{SP}(x)$ and $f_\delta = F(\eta_\delta, v_\delta, l_\delta)$ and using the dynamic programming principle, we find that

$$\begin{aligned} u(x, t) &\leq \int_0^S (L(\eta_\delta(s), -v_\delta(s)) + f_\delta(s)) ds + u(\eta_\delta(S), t - S) \\ &= \int_0^t (L(\eta(s), -v(s)) + f(s)) ds + \theta\omega_0(\delta) + u(y, T) \\ &< u_\infty(x) + T_0\omega_0(\delta) + \varepsilon, \end{aligned}$$

which implies that $\limsup_{t \rightarrow \infty} u(x, t) \leq u_\infty(x)$. The proof is now complete. \square

Finally we briefly sketch some arguments in order to explain how to adapt the dynamical approach to Theorem 1.5 for (CN) to that of Theorem 1.5 for (DBC). As usual, we assume that $c_* = 0$ and let u_∞ be the same function as in (3.24). We infer that $\liminf_{t \rightarrow \infty} u(x, t) = u_\infty(x)$ for all $x \in \bar{\Omega}$. To prove the inequality $\limsup_{t \rightarrow \infty} u(x, t) \leq u_\infty(x)$ for $x \in \bar{\Omega}$, we fix $x \in \bar{\Omega}$ and $\varepsilon > 0$, and select a triple $(\eta, v, l) \in \text{SP}(x)$, a constant $T_0 > 0$ and a family $\{T(y)\}_{y \in \bar{\Omega}} \subset (0, T_0]$ as in the dynamical approach.

We fix $t > 0$ and introduce the function τ on $[0, \infty)$ given by $\tau(s) = t - \int_0^s (1 + l(r)) dr$. Define the constant $\sigma \in (0, t]$ by $\tau(\sigma) = 0$. Since $l \in L^\infty([0, \infty), \mathbb{R})$, we have $\sigma \rightarrow \infty$ as $t \rightarrow \infty$. We set $y = \eta(\sigma)$, $T = T(y)$, $S = \sigma - T$ and $\delta = (\sigma - S)/S$. We note that $\delta S = \sigma - S = T$ and that, as $t \rightarrow \infty$, $S = \sigma - T \rightarrow \infty$ and $\delta = T/S \rightarrow 0$. We assume henceforth that t is large enough so that $\delta < \delta_0$.

We define $\eta_\delta, v_\delta, l_\delta$ and f_δ as in the dynamical approach for (CN). We define the function τ_δ on $[0, \infty)$ by $\tau_\delta(s) = t - \int_0^s (1 + l_\delta(r)) dr$ for $s \geq 0$. It is easily seen that $\tau_\delta(S) = T$. Then we compute similarly that

$$\int_0^S L(\eta_\delta(s), -v_\delta(s)) ds \leq \int_0^\sigma L(\eta(s), -v(s)) ds + \delta S \omega_0(\delta),$$

where $\omega_0 \in C([0, \infty))$ is a function satisfying $\omega_0(0) = 0$, and

$$\begin{aligned} u(x, t) &\leq \int_0^S (L(\eta_\delta(s), -v_\delta(s)) + f_\delta(s)) ds + u(\eta_\delta(S), \tau_\delta(S)) \\ &\leq \int_0^\sigma (L(\eta(s), -v(s)) + f(s)) ds + T_0 \omega_0(\delta) + u(\eta(\sigma), T) \\ &< u_\infty(x) + T_0 \omega_0(\delta) + \varepsilon, \end{aligned}$$

from which we conclude that $\limsup_{t \rightarrow \infty} u(x, t) \leq u_\infty(x)$ for all $x \in \bar{\Omega}$.

3.5. A formula for u_∞ . Once the additive eigenvalues of (E1) and (E2) are normalized so that $c = 0$, the correspondence between the initial data u_0 and the asymptotic solution u_∞ is the same for both (CN) and (DBC).

We assume that $c = 0$, and present in this section another formula for the function u_∞ given by (3.24).

We introduce the Aubry (or, Aubry-Mather) set \mathcal{A} for (E1), with $a = 0$. We first define the function $d \in \text{Lip}(\bar{\Omega} \times \Omega)$ by

$$d(x, y) = \inf\{\psi(x) - \psi(y) : \psi \text{ is a subsolution of (E1), with } a = 0\},$$

and then the Aubry (or, Aubry-Mather) set \mathcal{A} for (E1), with $a = 0$, as the subset of $\bar{\Omega}$ consisting of those points y where the function $d(\cdot, y)$ is a solution of (E1), with $a = 0$.

Theorem 3.15. *The function u_∞ given by (3.24) is represented as*

$$u_\infty(x) = \inf\{d(x, y) + d(y, z) + u_0(z) : z \in \bar{\Omega}, y \in \mathcal{A}\}.$$

The function u_0^- has the formula similar to the above: $u_0^-(x) = \inf\{d(x, y) + u_0(y) : y \in \bar{\Omega}\}$. Accordingly, we have $u_\infty(x) = \inf\{d(x, y) + u_0^-(y) : y \in \mathcal{A}\}$.

We do not give the proof of these formulas, and instead we refer the reader to [23, Proposition 4.4] and [5, Proposition 6.3] where these formulas are established for (E1) with the linear Neumann condition.

4. APPENDIX

4.1. Construction of test-functions. In order to prove Theorem 1.1, we have to build test-functions. For the convenience of the reader, we briefly recall how to construct these functions and refer to [1, 2, 3, 20] for more general cases as well as more details.

Lemma 4.1. *There exists $M_1 > 0$ such that for any $\delta \in (0, 1)$ and $\xi \in \partial\Omega$, there exists a function $C^{\xi, \delta} \in C^1(\mathbb{R}^{n+1})$ such that*

$$\begin{aligned} |q + B(\xi, p + C^{\xi, \delta}(p, q)\tilde{n}(\xi))| &\leq m(\delta), \\ |D_p C^{\xi, \delta}(p, q)| + |D_q C^{\xi, \delta}(p, q)| &\leq M_1 \text{ and} \\ D_q C^{\xi, \delta}(p, q) &\leq 0 \end{aligned}$$

for any $(p, q) \in \mathbb{R}^n \times \mathbb{R}$, where m is a modulus.

Proof. By (A2) and (A3) there exists a function $C^\xi \in C(\mathbb{R}^n \times \mathbb{R})$ such that

$$\begin{aligned} q + B(\xi, p + C^\xi(p, q)\tilde{n}(\xi)) &= 0, \\ |C^\xi(p_1, q_1) - C^\xi(p_2, q_2)| &\leq M_1(|p_1 - p_2| + |q_1 - q_2|) \end{aligned}$$

for some $M_1 > 0$. Noting that $r \mapsto B(\xi, p + r\tilde{n}(\xi))$ is increasing, we see that $q \mapsto C^\xi(p, q)$ is decreasing for any $p \in \mathbb{R}^n$. Therefore we see that a regularized function $C^{\xi, \delta}$ by a mollification kernel satisfies the desired properties. \square

Similarly we can prove

Lemma 4.2. *For any $a > 0$, there exists $M_{1a} > 0$ such that for any $b \in \mathbb{R}$, $\delta \in (0, 1)$, and $\xi \in \partial\Omega$, there exists a function $C_{a,b}^{\xi, \delta} \in C^1(\mathbb{R}^n)$ such that*

$$\begin{aligned} |b + B(\xi, a(p + C_{a,b}^{\xi, \delta}(-p)\tilde{n}(\xi)))| &\leq m_a(\delta), \\ |DC_{a,b}^{\xi, \delta}(p)| &\leq M_{1a} \end{aligned}$$

for any $p \in \mathbb{R}^n$, where m_a is a modulus.

Now we are in position to build the test-functions we need. We are going to do it locally, i.e. in a neighborhood of a point $\xi \in \partial\Omega$ and we set $\rho_\xi(x) := \tilde{n}(\xi) \cdot x$.

Lemma 4.3. *For fixed constants $a, b \in \mathbb{R}$, we denote by $C_{a,b}^{\xi, \delta}$, m_a , and M_{1a} the functions and the constant given in Lemma 4.2 for $\delta \in (0, 1)$ and $\xi \in \partial\Omega$. We introduce the function χ defined, for $Z \in \mathbb{R}^n$, by*

$$\chi(Z) := \frac{|Z|^2}{2\varepsilon^2} - C_{a,b}^{\xi, \delta}\left(\frac{Z}{\varepsilon^2}\right)\rho_\xi(Z) + \frac{A(\rho_\xi(Z))^2}{\varepsilon^2}$$

for $\varepsilon \in (0, 1)$ and $A > 0$. If $A \geq \max\{M_{1a}^2, (M_{1a}M_B)/2\theta\} =: M_{2a}$, then

- (i) $\chi(Z) \geq \frac{|Z|^2}{4\varepsilon^2} - M_{1a}|Z|$ for all $Z \in \mathbb{R}^n$,
(ii) For all $R > 0$ there exists a modulus $m = m_{R,a}$ such that if $|x-y|/\varepsilon^2 \leq R$, then

$$b + B(x, -aD\chi(x-y)) \leq m(\delta + \varepsilon + |x - \xi| + |y - \xi|),$$

if $x \in \partial\Omega, y \in \overline{\Omega}$ and

$$b + B(y, -aD\chi(x-y)) \geq -m(\delta + \varepsilon + |x - \xi| + |y - \xi|),$$

for $x \in \overline{\Omega}, y \in \partial\Omega$.

Proof. We first prove (i). Note that $C_{a,b}^{\xi,\delta}(p) \leq C_{a,b}^{\xi,\delta}(0) + M_{1a}|p|$ for all $p \in \mathbb{R}^n$ by Lemma 4.2. Thus,

$$\begin{aligned} \chi(Z) &\geq \frac{|Z|^2}{2\varepsilon^2} - M_{1a}\left(\frac{|Z|}{\varepsilon^2} + 1\right)|\rho_\xi(Z)| + \frac{A(\rho_\xi(Z))^2}{\varepsilon^2} \\ &\geq \frac{|Z|^2}{4\varepsilon^2} - M_{1a}|Z| + \frac{(\rho_\xi(Z))^2}{\varepsilon^2}(A - M_{1a}^2) \\ &\geq \frac{|Z|^2}{4\varepsilon^2} - M_{1a}|Z| \end{aligned}$$

for all $Z \in \mathbb{R}^n$. We have used Young's inequality in the second inequality above.

We next prove (ii). We have for any $x \in \partial\Omega$

$$\begin{aligned} -aD\chi(x-y) &= a\left(\frac{y-x}{\varepsilon^2} + C_{a,b}^{\xi,\delta}\left(\frac{x-y}{\varepsilon^2}\right)\tilde{n}(\xi)\right) \\ &\quad + a\frac{(\rho_\xi(x) - \rho_\xi(y))}{\varepsilon^2}\left(DC_{a,b}^{\xi,\delta}\left(\frac{x-y}{\varepsilon^2}\right) - 2A\tilde{n}(\xi)\right). \end{aligned}$$

We divide into two cases: (a) $\rho_\xi(x) - \rho_\xi(y) \leq 0$; (b) $\rho_\xi(x) - \rho_\xi(y) > 0$.

We first consider Case (a). Using (A0) and a Taylor expansion at the point $(x+y)/2$, it is easy to see that

$$0 \leq \rho(x) - \rho(y) = \tilde{n}((x+y)/2) \cdot (x-y) + o(|x-y|) \text{ for } x \in \partial\Omega.$$

Using the continuity of $D\rho$, and therefore of \tilde{n} , we see that

$$\begin{aligned} \rho_\xi(x) - \rho_\xi(y) &= \tilde{n}(\xi) \cdot (x-y) \\ &= \tilde{n}((x+y)/2) \cdot (x-y) + (\tilde{n}(\xi) - \tilde{n}((x+y)/2)) \cdot (x-y) \\ &\geq o(|x-y|) - m(|x-\xi| + |y-\xi|)|x-y|. \end{aligned}$$

for some modulus m . Therefore, taking into account the restriction $|x-y|/\varepsilon^2 \leq R$ and changing perhaps the modulus m , we have

$$\frac{\rho_\xi(x) - \rho_\xi(y)}{\varepsilon^2} \geq -m(\varepsilon + |x - \xi| + |y - \xi|),$$

and this yields

$$\frac{\rho_\xi(x) - \rho_\xi(y)}{\varepsilon^2} \rightarrow 0 \quad \text{and} \quad -a\frac{(\rho_\xi(x) - \rho_\xi(y))}{\varepsilon^2}\left(DC_{a,b}^{\xi,\delta}\left(\frac{x-y}{\varepsilon^2}\right) - 2A\tilde{n}(\xi)\right) \rightarrow 0$$

as $\varepsilon \rightarrow 0$, $x \rightarrow \xi$, and $y \rightarrow \xi$, which implies that

$$\begin{aligned} b + B(x, -aD\chi(x-y)) &\leq b + B(x, a\left(\frac{y-x}{\varepsilon^2} + C_{a,b}^{\xi,\delta}\left(\frac{x-y}{\varepsilon^2}\right)\tilde{n}(\xi)\right)) \\ &\quad + M_B \left| a \frac{(\rho_\xi(x) - \rho_\xi(y))}{\varepsilon^2} (DC_{a,b}^{\xi,\delta}\left(\frac{x-y}{\varepsilon^2}\right) - 2A\tilde{n}(\xi)) \right| \\ &\leq m(\delta + \varepsilon + |x - \xi| + |y - \xi|) \end{aligned}$$

In Case (b) by (A3), (A4) and Lemma 4.2, we get (changing perhaps the modulus m)

$$\begin{aligned} &b + B(x, -aD\chi(x-y)) \\ &\leq b + B(x, a\left(\frac{y-x}{\varepsilon^2} + C_{a,b}^{\xi,\delta}\left(\frac{x-y}{\varepsilon^2}\right)\tilde{n}(\xi)\right)) + a \frac{(\rho_\xi(x) - \rho_\xi(y))}{\varepsilon^2} (M_B |DC_{a,b}^{\xi,\delta}| - 2A\theta) \\ &\leq b + B(\xi, a\left(\frac{y-x}{\varepsilon^2} + C_{a,b}^{\xi,\delta}\left(\frac{x-y}{\varepsilon^2}\right)\tilde{n}(\xi)\right)) + m(|x - \xi|) \\ &\leq m(\delta + |x - \xi|), \end{aligned}$$

since $A \geq (M_{1a}M_B)/2\theta$. Gathering the two cases, we have the result. Similarly we obtain $b + B(y, -aD\chi(x-y)) \geq -m(\delta + \varepsilon + |x - \xi| + |y - \xi|)$ if $y \in \partial\Omega$. \square

4.2. Comparison Results for (CN) and (DBC).

Proof of Theorem 1.1. We argue by contradiction assuming that there would exist $T > 0$ such that $\max_{\overline{Q_T}}(u - v)(x, t) > 0$, where $Q_T := \Omega \times (0, T)$.

Let u^γ denote the function

$$u^\gamma(x, t) := \max_{s \in [0, T+2]} \{u(x, s) - (1/\gamma)(t - s)^2\},$$

for any $\gamma > 0$. This sup-convolution procedure is standard in the theory of viscosity solutions (although, here, it acts only on the time-variable) and it is known that, for γ small enough, u^γ is a subsolution of (CN) in $\Omega \times (a_\gamma, T+1)$, where $a_\gamma := (2\gamma \max_{Q_{T+2}} |u(x, t)|)^{1/2}$ (see [2, 9] for instance).

Moreover, it is easy to check that $|u_t^\gamma| \leq M_\gamma$ in $\Omega \times (a_\gamma, T+1)$ and therefore by the coercivity of H and the C^1 -regularity of $\partial\Omega$ we have, for all $x, y \in \overline{\Omega}$, $t, s \in [a_\gamma, T+1]$

$$|u^\gamma(x, t) - u^\gamma(y, s)| \leq M_\gamma(|x - y| + |t - s|) \quad (4.1)$$

for some $M_\gamma > 0$. Finally, as $\gamma \rightarrow 0$, $\max_{\overline{Q_T}}(u^\gamma - v)(x, t) \rightarrow \max_{\overline{Q_T}}(u - v)(x, t) > 0$.

Therefore it is enough to consider $\max_{\overline{Q_T}}(u^\gamma - v)(x, t)$ for $\gamma > 0$ small enough, and we follow the classical proof by introducing

$$\max_{\overline{Q_T}} \{(u^\gamma - v)(x, t) - \eta t\},$$

for $0 < \eta \ll 1$. This maximum is achieved at $(\xi, \tau) \in \overline{\Omega} \times [0, T]$, namely $(u^\gamma - v)(\xi, \tau) - \eta\tau = \max_{\overline{Q_T}} \{(u^\gamma - v)(x, t) - \eta t\}$. Clearly τ depends on η but we can assume that it remains bounded away from 0, otherwise we easily get a contradiction.

We only consider the case where $\xi \in \partial\Omega$. We first consider problem (CN). We introduce the function χ_1 defined by : $\chi_1(Z) := \chi(-Z)$ where χ is the function given by Lemma 4.3 with $a = 1$ and $b = 0$. It is worth pointing out that, compared to the proof of Lemma 2.2, the change $Z \rightarrow -Z$, consists in exchanging the role of x and y , which is natural since the variable x is used here for the subsolution while it was corresponding to a supersolution in the proof of Lemma 2.2.

We define the function $\Psi : \overline{\Omega}^2 \times [0, T] \rightarrow \mathbb{R}$ by

$$\begin{aligned} \Psi_1(x, y, t) := & u^\gamma(x, t) - v(y, t) - \eta t - \chi_1(x - y) - \alpha(\rho(x) + \rho(y)) \\ & - |x - \xi|^2 - (t - \tau)^2. \end{aligned}$$

Let Ψ_1 achieve its maximum at $(\bar{x}, \bar{y}, \bar{t}) \in \overline{\Omega}^2 \times [0, T]$. By standard arguments, we have

$$\bar{x}, \bar{y} \rightarrow \xi \text{ and } \bar{t} \rightarrow \tau \text{ as } \varepsilon \rightarrow 0 \quad (4.2)$$

by taking a subsequence if necessary. In view of the Lipschitz continuity (4.1) of u^γ , we have

$$|\bar{p}| \leq M_\gamma, \quad (4.3)$$

where $\bar{p} := (\bar{x} - \bar{y})/\varepsilon^2$.

Taking (formally) the derivative of Ψ_1 with respect to each variable x, y at $(\bar{x}, \bar{y}, \bar{t})$, we have

$$\begin{aligned} D_x u^\gamma(\bar{x}, \bar{t}) &= D_x \chi_1(\bar{x} - \bar{y}) + 2(\bar{x} - \xi) + \alpha \tilde{n}(\bar{x}), \\ D_y v(\bar{y}, \bar{t}) &= D_y \chi_1(\bar{x} - \bar{y}) - \alpha \tilde{n}(\bar{y}). \end{aligned}$$

We remark that we should interpret $D_x u^\gamma$ and $D_y v$ in the viscosity solution sense here. We also point out that the viscosity inequalities we are going to write down below, hold up to time T , in the spirit of [2], Lemma 2.8, p. 41.

By Lemma 4.3 we obtain

$$\begin{aligned} B(\bar{x}, D_x u(\bar{x}, \bar{t})) &\geq -m(\delta + \varepsilon + |\bar{x} - \xi| + |\bar{y} - \xi|) + \theta\alpha > 0, \\ B(\bar{y}, D_y v(\bar{y}, \bar{t})) &\leq m(\delta + \varepsilon + |\bar{x} - \xi| + |\bar{y} - \xi|) - \theta\alpha < 0 \end{aligned}$$

for $\varepsilon, \delta > 0$ which are small enough compared to $\alpha > 0$, where m is a modulus.

Therefore, by the definition of viscosity solutions of (CN), using the arguments of User's guide to viscosity solutions [9], there exists $a_1, a_2 \in \mathbb{R}$ such that

$$\begin{aligned} a_1 + H(\bar{x}, D_x u^\gamma(\bar{x}, \bar{t})) &\leq 0, \\ a_2 + H(\bar{y}, D_y v(\bar{y}, \bar{t})) &\geq 0 \end{aligned}$$

with $a_1 - a_2 = \eta + 2(\bar{t} - \tau)$. By (4.3) we may assume that $\bar{p} \rightarrow p$ as $\varepsilon \rightarrow 0$ for some $p \in \mathbb{R}^n$ by taking a subsequence if necessary. Sending $\varepsilon \rightarrow 0$ and then $\alpha \rightarrow 0$ in the above inequalities, we have a contradiction since $a_1 - a_2 \rightarrow \eta > 0$ while the H -terms converge to the same limit. Therefore τ cannot be assumed to remain bounded away from 0 and the conclusion follows.

We next consider problem (DBC). Let (ξ, τ) be defined as above and let $C^{\xi, \delta}$ be the function given by Lemma 4.1. We define the function $\chi_2 : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} \chi_2(x - y, t - s) := & \frac{1}{2\varepsilon^2} \left(|x - y|^2 + (t - s)^2 \right) + C^{\xi, \delta} \left(\frac{x - y}{\varepsilon^2}, \frac{t - s}{\varepsilon^2} \right) (\rho(x) - \rho(y)) \\ & + \frac{A(\rho(x) - \rho(y))^2}{\varepsilon^2} \end{aligned}$$

for $A \geq M_1^2$. We define the function $\Psi : \overline{\Omega}^2 \times [0, T] \rightarrow \mathbb{R}$ by

$$\begin{aligned} \Psi_2(x, y, t, s) := & u^\gamma(x, t) - v(y, s) - \eta t - \chi_2(x - y, t - s) + \alpha(\rho(x) + \rho(y)) \\ & - |x - \xi|^2 - (t - \tau)^2. \end{aligned}$$

Let Ψ_2 achieve its maximum at $(\bar{x}, \bar{y}, \bar{t}, \bar{s}) \in \overline{\Omega}^2 \times [0, T]^2$ and set

$$\bar{p} := \frac{\bar{x} - \bar{y}}{\varepsilon^2}, \quad \bar{q} := \frac{\bar{t} - \bar{s}}{\varepsilon^2}.$$

Derivating (formally) Ψ_2 with respect to each variable t, s at $(\bar{x}, \bar{y}, \bar{t}, \bar{s})$, we have

$$\begin{aligned} u_t^\gamma(\bar{x}, \bar{t}) &= \eta + \bar{q} + D_q C^{\xi, \delta}(\bar{p}, \bar{q}) \cdot \frac{(\rho(\bar{x}) - \rho(\bar{y}))}{\varepsilon^2}, \\ v_s(\bar{y}, \bar{s}) &= \bar{q} + D_q C^{\xi, \delta}(\bar{p}, \bar{q}) \cdot \frac{(\rho(\bar{x}) - \rho(\bar{y}))}{\varepsilon^2}. \end{aligned}$$

We remark that we should interpret u_t^γ and v_s in the viscosity solution sense here.

We consider the case where $\bar{x} \in \partial\Omega$. Note that $D_q C^{\xi, \delta}(\bar{p}, \bar{q}) \leq 0$ and then we have

$$\begin{aligned} & u_t^\gamma(\bar{x}, \bar{t}) + B(\bar{x}, D_x u(\bar{x}, \bar{t})) \\ &= \eta - D_q C^{\xi, \delta}(\bar{p}, \bar{q}) \cdot \frac{\rho(\bar{y})}{\varepsilon^2} + \bar{q} + B(\bar{x}, D_x u(\bar{x}, \bar{t})) \\ &\geq -m(\delta + \varepsilon + |\bar{x} - \xi| + |\bar{y} - \xi|) + \theta\alpha > 0 \end{aligned}$$

for $\varepsilon, \delta > 0$ which are small enough compared to $\alpha > 0$. In the case where $\bar{y} \in \partial\Omega$ we similarly obtain

$$v_t(\bar{y}, \bar{s}) + B(\bar{y}, D_y v(\bar{y})) \leq m(\delta + \varepsilon) - \theta\alpha < 0$$

for $\varepsilon, \delta > 0$ which are small enough compared to $\alpha > 0$.

The rest of the argument is similar to that given above and therefore we omit the details here. \square

4.3. Existence and Regularity of Solutions of (CN) and (DBC).

Proof of Theorem 1.2. The existence part being standard by using the Perron's method (see [19]), we mainly concentrate on the regularity of solutions when $u_0 \in W^{1, \infty}(\overline{\Omega})$. We may choose a sequence $\{u_0^k\}_{k \in \mathbb{N}} \subset C^1(\overline{\Omega})$ so that $\|u_0^k - u_0\|_{L^\infty(\Omega)} \leq 1/k$ and $\|Du_0^k\|_\infty \leq C$ for some $C > 0$ which is uniform for all $k \in \mathbb{N}$. We fix $k \in \mathbb{N}$.

We claim that $u_-^k(x, t) := -M_1 t + u_0^k(x)$ and $u_+^k(x, t) := M_1 t + u_0^k(x)$ are, respectively, a sub and supersolution of (CN) or (DBC) with $u_0 = u_0^k$ for a suitable large $M_1 > 0$. We can easily see that u_\pm^k are a sub and supersolution of (CN) or (DBC) in Ω if $M_1 \geq \max\{|H(x, p)| : x \in \overline{\Omega}, p \in B(0, C)\}$.

We recall (see [25, 9] for instance) that if $x \in \partial\Omega$, then

$$D^+ u_-^k(x, t) = \{(Du_0^k(x) + \lambda \tilde{n}(x), -M_1) : \lambda \leq 0\},$$

where $D^+ u_-^k(x, t)$ denotes the super-differential of u_-^k at (x, t) . We need to show that

$$\min\{-M_1 + H(x, Du_0^k(x) + \lambda \tilde{n}(x)), B(x, Du_0^k(x) + \lambda \tilde{n}(x))\} \leq 0$$

for all $\lambda \leq 0$. By (A2) it is clear enough that there exists $\bar{\lambda} < 0$ such that, if $\lambda \leq \bar{\lambda}$, then $B(x, Du_0^k(x) + \lambda \tilde{n}(x)) \leq 0$. Then choosing $M_1 \geq \max\{H(x, p + \lambda \tilde{n}(x)) : x \in \partial\Omega, p \in B(0, C), \bar{\lambda} \leq \lambda \leq 0\}$, the above inequality holds. A similar argument shows that u_+^k is a supersolution of (CN) for M_1 large enough. It is worth pointing out that such M_1 is independent of k . We can easily check that u_\pm^k are a sub and supersolution of (DBC) on $\partial\Omega \times (0, \infty)$ too.

By Perron's method (see [19]) and Theorem 1.1, we obtain continuous solutions of (CN) or (DBC) with $u_0 = u_0^k$ that we denote by u^k . As a consequence of Perron's method, we have

$$-M_1 t + u_0^k(x) \leq u^k(x, t) \leq M_1 t + u_0^k(x) \text{ on } \overline{\Omega} \times [0, \infty).$$

To conclude, we use a standard argument: comparing the solutions $u^k(x, t)$ and $u^k(x, t + h)$ for some $h > 0$ and using the above property on the u^k , we have

$$\|u^k(\cdot, \cdot + h) - u^k(\cdot, \cdot)\|_\infty \leq \|u^k(\cdot, h) - u^k(\cdot, 0)\|_\infty \leq M_1 h.$$

As a consequence we have $\|(u^k)_t\|_\infty \leq M_1$ and, by using the equation together with (A1), we obtain that Du^k is also bounded. Finally sending $k \rightarrow \infty$ by taking a subsequence if necessary we obtain the Lipschitz continuous solution of (CN) or (DBC).

We finally remark that, if $u_0 \in C(\overline{\Omega})$, we can obtain the existence of the uniformly continuous solution on $\overline{\Omega} \times [0, \infty)$ by using the above result for $u_0 \in W^{1,\infty}(\overline{\Omega})$ and (1.1) which is a direct consequence of Theorem 1.1. \square

4.4. Additive Eigenvalue Problems.

Proof of Theorem 1.3. We first prove (i). For any $\varepsilon \in (0, 1)$ we consider

$$\begin{cases} \varepsilon u_\varepsilon + H(x, Du_\varepsilon) = 0 & \text{in } \Omega, \\ B(x, Du_\varepsilon) = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4)$$

Following similar arguments as in the proof of Theorem 1.2, it is easy to prove that, for $C > 0$ large enough $-C/\varepsilon$ and C/ε are, respectively, a subsolution and a supersolution of (4.4) or (4.6).

We remark that, because of (A1) and the regularity of the boundary of Ω , the subsolutions w of (4.4) such that $-C/\varepsilon \leq w \leq C/\varepsilon$ on $\overline{\Omega}$ satisfy $|Dw| \leq M_2$ in Ω for some $M_2 > 0$ and therefore they are equi-Lipschitz continuous on

$\overline{\Omega}$. With these informations, Perron's method provides us with a solution $u_\varepsilon \in W^{1,\infty}(\Omega)$ of (4.4). Moreover, by construction, we have

$$|\varepsilon u_\varepsilon| \leq M_1 \text{ on } \overline{\Omega} \quad \text{and} \quad |Du_\varepsilon| \leq M_2 \text{ in } \Omega. \quad (4.5)$$

Next we set $v_\varepsilon(x) := u_\varepsilon(x) - u_\varepsilon(x_0)$ for a fixed $x_0 \in \overline{\Omega}$. Because of (4.5) and the regularity of the boundary $\partial\Omega$, $\{v_\varepsilon\}_{\varepsilon \in (0,1)}$ is a sequence of equi-Lipschitz continuous and uniformly bounded functions on $\overline{\Omega}$. By Ascoli-Arzelà's Theorem, there exist subsequences $\{v_{\varepsilon_j}\}_j$ and $\{u_{\varepsilon_j}\}_j$ such that

$$v_{\varepsilon_j} \rightarrow v, \quad \varepsilon_j u_{\varepsilon_j} \rightarrow -c \text{ uniformly on } \overline{\Omega}$$

as $j \rightarrow \infty$ for some $v \in W^{1,\infty}(\Omega)$ and $c \in \mathbb{R}$. By a standard stability result of viscosity solutions we see that (v, c) is a solution of (E1).

In order to prove (ii) we just need to consider

$$\begin{cases} \varepsilon u_\varepsilon + H(x, Du_\varepsilon) = 0 & \text{in } \Omega, \\ \varepsilon u_\varepsilon + B(x, Du_\varepsilon) = 0 & \text{on } \partial\Omega \end{cases} \quad (4.6)$$

instead of (4.4). By the same argument above we obtain a solution of (E2). \square

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